Solution to Chapter 2's problems

Q1. For the southward facing side, the angle δ between the normal to the slope and the beam is $\delta = \theta_S - \alpha$. The intensity reaching the slope is then $I \cos\delta = I\cos(\alpha - \theta_S)$ (notice that it makes sense, no light reaching the slope when it is parallel to the slope, i.e., when $\theta_S = \pi/2 - \alpha$). For the northward facing side, $\delta = \alpha + \theta_S$ so that the ratio sought is $\cos(\alpha - \theta_S)/\cos(\alpha + \theta_S)$. This is 1.53 for $\theta_S = 30°$ and 4.4 for $\theta_S = 60°$.

Q2. The irradiance and intensity are related through $F = \int I \cos\theta d\Omega$. Here we consider the case of zero zenith angle so $\theta = 0$. The rays reaching the TOA do not originate equally from all directions. Rather they originate from a "radiation pencil" whose solid angle is $\delta\Omega$ (Fig. 1). The fraction of the hemisphere of solid angle (i.e., "the sky") that is occupied by the Sun is the same as the fraction of the area of the hemisphere of radius d, centered on Earth, occupied by the Sun: $\delta\Omega/4\pi = \pi R_s^2/(4\pi d^2) = 6.84 \times 10^{-5} sr^{-1}$. Hence, $I \approx F/\delta\Omega = 2 \times 10^7 W m^{-2} sr^{-1}$.

NB: Alternatively, you could work out the solid angle from the definition $\delta\Omega = \int_0^\theta \sin\theta d\theta d\phi$ in which θ now measures the angle between the vertical and the Sun's surface at its equator, and ϕ is the azimuthal angle (from 0 to 2π). From the geometry of the problem, $\sin\theta = R_s/\sqrt{R_s^2 + d^2} \approx R_s/d$ so that, approximating the integral as half the area of the rectangle with height $\sin\theta$ and width θ , we get $\delta\Omega = 2\pi \times R_s/d \times \frac{1}{2} R_s/d = \pi(R_s/d)^2$, as before.

Q3. From Beer's law, the ratio of radiation intensity is $e^{-\beta_\lambda \int_0^s \rho_a ds}$. (i) We have $\int_0^s \rho_a ds = 1 kg m^{-2}$ so the ratio is $e^{-0.01\times 1} = 0.99$, i.e., very little absorption. (ii) Just rearrange Beer's law to get $\int_0^s \rho_a ds = \ln 2/\beta_\lambda = 69.3 kg m^{-2}$.

Q4. (i) From the definition of optical depth, $\tau(z) = \int_z^{+\infty} \rho q k_\lambda dz$ with $k_\lambda = \alpha_\lambda = 0.01 m^2 kg^{-1}$ (from the previous question). Using the hydrostatic equation, $dP/dz = -\rho g$, and noticing that k_λ and q are constants, we get, $\tau(z) = q k_\lambda \int_0^{P_s} dP/g = q k_\lambda P/g$. (ii) To reach an optical depth of unity, P must be equal to $g/(q k_\lambda) = 9.81/(0.01 \times 10^{-3}) = 9.81.10^5 Pa$. This is about ten times larger than P_s so cannot be reached. The optical depth increases with pressure, with maximum value at the surface, $\tau(z = 0) = q k_\lambda P_s/g = 0.1$.

Q5. Write $\rho = \rho_s e^{-z/H_s}$ in which H_s is the scale height and ρ_s the surface

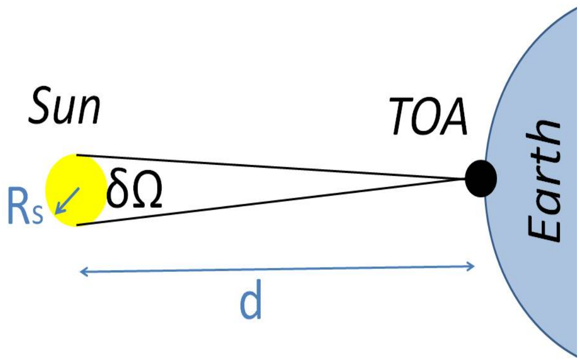

Figure 1: Approximate calculation of solid angle for Q2.

density. From the definition of optical depth,

$$\tau_\lambda(z) \equiv \int_z^{+\infty} \rho q_a k_\lambda dz = q_a k_\lambda \rho_s H_s e^{-z/H_s} \tag{1}$$

assuming uniform mixing ratio q_a. From the notes, $Q_\lambda \propto \rho F_\lambda^\downarrow \propto \rho e^{-\tau(z)}$. Hence the heating rate is maximum when

$$\frac{dQ_\lambda}{dz} = 0 \quad \text{i.e.,} \; \frac{d\rho}{dz} e^{-\tau} = \rho \frac{d\tau}{dz} e^{-\tau} \tag{2}$$

This can be rewritten as $1 = q_a k_\lambda \rho_s H_s e^{-z/H_s} = \tau$, hence the result.

Q6. Simply divide the previous answer by ρ to express the heating rate in $W kg^{-1}$ rather than in $W m^{-3}$,

$$Q_\lambda / \rho \propto e^{-\tau_\lambda(z)} \tag{3}$$

which is indeed maximum at the top-of-the-atmosphere. This simply reflects the fact that, per unit mass, there will be more absorption of radiation at upper levels (more intensity of radiation incoming) than at low levels (not much radiation left to absorb).

Q7. (i) Denote by I_o the intensity emitted by the surface, I_1 that impinging at the height where $\tau = 0.2$ and I_2 the intensity at the height where $\tau = 4$. What we are asked to compute is $(I_1 - I_2)/I_o$. Using Beer's law (or the first term on the right hand side in Schwarzchild equation), this is simply $e^{-0.2} - e^{-4} = 0.8$, i.e., 80 % of the radiation emitted by the surface is absorbed by this layer. (ii) Denote by B_o the Planck function for the atmosphere and Earth's surface at temperature T_o, i.e., $B_o = B_\lambda(T_o)$. The total infrared radiation is obtained from Schwarzchild equation. At a fixed path distance s, with $s_o = 0$ (the Earth's surface), we have:

$$I_\lambda(s) = B_o e^{-\tau(0,s)} + \int_0^{\tau(0,s)} B_o e^{-(\tau(0,s)-\tau')} d\tau' = B_o \tag{4}$$

This shows that for the special case considered (isothermal atmosphere at the same temperature as the Earth's surface), the upward infrared flux is constant with height. Thus at the TOA, $I_\lambda = B_o$. The contribution ΔI_λ to the OLR from the layer sandwiched between $\tau(0,s) = 0.2$ and $\tau(0,s) = 4$ is simply,

$$\Delta I_\lambda = \int_{0.2}^{4} B_o e^{-(\tau_\infty - \tau')} d\tau' = B_o e^{-\tau_\infty}(e^4 - e^{0.2}) \approx 0.36 B_o \tag{5}$$

where we have used $\tau_\infty = 5$. The contribution is thus $0.36B_o/B_o = 36$ %.

Q8.

(i) The radiative balance is $4\pi R^2 \sigma T_e^4 = \pi R^2 (1-\alpha_P) S_o$ so $T_e = ((1-\alpha_P)S_o/4\sigma)^{1/4}$.

(ii) Differentiating the radiative balance with respect to T_e, one obtains, $16\sigma T_e^3 \delta T_e = -S_o \delta\alpha_P$ where $\delta\alpha_P$ is the change in albedo and δT_e the resulting change in emission temperature. Rearranging, one gets $\delta T_e / T_e = \delta(1-\alpha_P)/4(1-\alpha_P)$. This shows that a 10 % change in $(1-\alpha_P)$ leads to a $10/4 = 2.5$ % change in emission temperature. Expressed as $\delta T_e/\delta\alpha_P$ the sensitivity is $-91K$ per unit change in planetary albedo. The negative sign reflects that a more reflective planet is colder.

(iii) Assuming $\delta T_s \approx \delta T_e$ where $\delta T_s = 0.2K$ is the change in global surface temperature, one gets $\delta T_s \approx (\delta T_e/\delta\alpha_P)\delta\alpha_P$. Using $\alpha_P = f\alpha_c + (1-f)\alpha_s$ in which f is the fraction of the Earth's surface covered by clouds and $\alpha_c = 0.8, \alpha_s = 0.1$, one then obtains, $\delta T_s \approx (\delta T_e/\delta\alpha_P)(\alpha_c - \alpha_s)\delta f$, or $\delta f = \delta T_e / [(\delta T_e/\delta\alpha_P)(\alpha_c - \alpha_s)] = 0.2/(-91 \times (0.8 - 0.1)] = -0.003$. The negative sign is consistent with a reduction in cloud cover leading to a surface warming.

(iv) The question is a bit unclear because it does not specify the wavelength (shortwave or longwave) of the transmissivity increase. Considering the previous questions, I am assuming the shortwave is considered here, and I am interpreting the increase in transmissivity by 10 % as a decrease in cloud albedo by 10 %. This corresponds to a change in planetary albedo of $\delta\alpha_P = f\delta\alpha_c = -0.1 f\alpha_c$. To estimate f, use $\alpha_P = f\alpha_c + (1-f)\alpha_s$, leading to $f = (\alpha_P - \alpha_s)/(\alpha_c - \alpha_s) = 0.2/0.7 = 0.28$. As a result, $\delta\alpha_P = -0.1 \times 0.28 \times 0.8 = 0.022$. The change in surface temperature is then, assuming again $\delta T_s \approx \delta T_e = (\delta T_e/\delta\alpha_P)\delta\alpha_P = -91 \times (-0.02) = +2K$.

(v) The reason why the observed surface temperature larger than the effective temperature is the greenhouse effect. The transmittance associated with solar radiation is larger than the transmittance for thermal radiation. This means that the atmosphere allows solar radiation to heat the surface but traps thermal radiation emitted by the surface, which yields further heating of the surface.

(vi) The Earth's surface cools by evaporation and also because it is warmer than the air above (thermals). As a result, there must be a net downward radiative flux (surface energy gained) to oppose this cooling if an equilibrium is to be observed.

Q9. The total energy radiated at the Sun's surface is $4\pi r_{sun}^2 \sigma T_{sun}^4$. At a distance of $1AU$ the same total energy is radiated by the Sun, neglecting the weak absorption of shortwave radiation as it travels through space. So $4\pi r_{sun}^2 \sigma T_{sun}^4 = 4\pi d^2 S_o$, with $d = 1AU$. Plugging numbers one finds $S_o = 1362 W m^{-2}$. This is indeed close to the observed values displayed in the figure.

Q10. (i) From $I_\lambda = I_{\lambda,\infty} e^{-\tau_\lambda \sec\theta}$, one gets $(I_\lambda)_1/(I_\lambda)_2 = e^{-\tau_\lambda(\sec\theta_1 - \sec\theta_2)}$. The answer follows after taking the logarithm and rearranging. (ii) If we make the assumption that the atmosphere is not changing much in terms of k_λ, ρ and q_a between the two solar paths considered, then τ_λ can be interpreted as a measure of total optical depth for the atmosphere. Plugging numbers, one gets $\tau_\lambda = \ln 1.12/(\sec 40 - \sec 20) = 0.47$.