Answers to Problems for Chapter 3

Q1

(i) Start from the definition of entropy for dry air, yielding $ds = c_{p,d}dT/T - R_d dP/P$. The atmosphere being in hydrostatic balance, $\alpha dP/dz = -g$ where α is the specific volume. Hence $ds = c_{p,d}dT/T + gR_d dz/(\alpha P)$. Using the ideal gas law, $P\alpha = R_d T$, this is also, $ds = c_{p,d}dT/T + gdz/T$. For the adiabatic case, $ds = 0$, so $c_{p,d}dT + gdz = 0$. The result follows. Numerically, $\Gamma_d = 9.81/1005 \approx 9.8K/km$.

(ii) Starting from the definition of potential temperature $\theta = T(P_{ref}/P)^{R_d/c_{p,d}}$, and differentiating the ln of this expression with respect to z, we get $\frac{T}{\theta}\partial\theta/\partial z = \partial T/\partial z - \frac{R_d T}{P c_{p,d}}\partial P/\partial z$. Using the hydrostatic equation $\partial P/\partial z = -\rho g$ and the ideal gas law $P = \rho R_d T$, this can be rewritten as, $\frac{T}{\theta}\partial\theta/\partial z = \partial T/\partial z + g/c_{p,d}$. The result follows.

(iii) A dry atmosphere with $N^2 \propto \partial\theta/\partial z < 0$ is unstable to vertical displacements. Thus, it might happen that sporadically the lapse is larger than Γ_d (i.e., the temperature decreases with height more rapidly than about $9.8K/km$) but the atmosphere would then quickly overturn and reach $\Gamma = \Gamma_d$. The opposite case $N^2 > 0$ is stable and associated with a lapse rate weaker than the dry adiabatic value. In the stratosphere absorption of ultraviolet radiation from the Sun leads to high entropies there and a lapse rate even opposite in sign to Γ_d!

Q2 The parcel undergoes a moist adiabatic ascent over $4km$. Its temperature thus decreases by $4 \times 6.5 = 26K$. It then experiences a dry adiabatic compression over $3km$ and thus an increase in temperature of $3 \times 10 = 30K$. The parcel is thus warmer by $4K$ on the plateau than when it started above the sea.

Q3 (i) At room temperature and pressure the phase diagram of water shows that only the vapour phase should exist at equilibrium. Thus, if there is no liquid water in the room at all, thermodynamic equilibrium has been reached. If however there is liquid water (say someone has just mopped the floor), then the situation is not in thermodynamic equilibrium: evaporation will occur until all the liquid water remains in the vapour phase (when that occurs, thermodynamic equilibrium has been restored). (ii) Not in thermodynamic equilibrium: the rain will evaporate into the dry air mass.

Q4. At the top-of-the-atmosphere, the equilibrium condition is: $\sigma T_e^4 = \sigma T_2^4$, hence $T_2 = T_e = 255K$. For layer 1, equilibrium requires $\sigma T_2^4 + \sigma T_s^4 = 2\sigma T_1^4$, while for the surface, it reads $\sigma T_e^4 + \sigma T_1^4 = \sigma T_s^4$ (you can read these two equations as simply saying that heating equals cooling). Substracting the previous two equations lead to $\sigma T_1^4 = 2\sigma T_e^4$ hence $T_1 = 2^{1/4} T_e = 303K$. Replacing this value in any of the two equilibrium conditions above then yield $T_s = 3^{1/4} T_e = 336K$.

Q5.

(i) Using a reference pressure $P_o = 1000hPa$, the potential temperatures are $\theta_1 = 335K$ (at $700hPa$ and using $T_1 = 303K$), and $\theta_2 = 331K$ (at $400hPa$ and using $T_2 = 255K$), while, at the surface $\theta_s = T_s = 336K$. This is clearly an unstable situation, with high entropy air below low entropy air, from the surface to upper levels.

(ii) Convection carries heat upwards. So the surface temperature should go down.

(iii) Consideration of the TOA radiative budget leads again to $T_2 = T_e$. Hence $T_1 = T_2 + \Delta T$ and $T_s = T_e + 2\Delta T$.

(iv) The surface energy balance is $\sigma T_s^4 + F_s = \sigma T_1^4 + \sigma T_e^4$ while energy conservation for layer 2 reads: $\sigma T_1^4 + F_c = 2\sigma T_2^4$. Using the expressions in (iii), we obtain: $F_s = \sigma T_e^4[1 + (1+x)^4 - (1+2x)^4]$ and $F_c = \sigma T_e^4[2 - (1+x)^4]$.

(v) The pressure difference between the layers is $300hPa$ which is about $3km$ thick. For a moist adiabat, the temperature decreases with height at about $7K/km$, hence a plausible choice for ΔT is $3 \times 7 = 21K$. This has a value $x = 21/255 = 0.08$. (NB: at constant entropy, temperature always increases with pressure so ΔT, and as a result x, must be positive). A plot of F_s and F_c as a function of x shows that these are positive and decrease with increasing x, up to a point where they cross the zero line. The solution physically requires that convection carries heat upward so that both F_s and F_c must be positive. This will be satisfied as long as $0 < x < 0.14$ (the zero crossing with the smallest value of x).

Finally, without convective fluxes, the surface temperature was $T_s = 3^{1/4} T_e$. With convective fluxes, $T_s = T_e(1 + 2x)$ and so equals the radiative equilibrum value when $x = (3^{1/4} - 1)/2 = 0.158$. Thus, over the range of values for the which is physical ($0 < x < 0.14$, based on $F_s, F_c > 0$) the surface temperature is smaller than the radiative equilibrium value, as expected from (ii).

Q6. It is just a matter of finding values for N^2, Δy and μ_θ. Inspection of the θ distribution from the ERA40 atlas suggests (at $45°N$) that $N^2 \approx (g/285K)20K/400hPa \approx (g/285)(20/4km) = 1.7 \times 10^{-4}s^{-2}$. At the same latitude, following the $\theta = 285K$ surface, one gets $\mu_\theta \approx 4km/5000km = 8 \times 10^{-4}$. The latitudinal extent of the storm can be obtained from the global infrared picture in Chapter 1, typically $\Delta y = 15°$ of latitude, i.e., $\Delta y = 6371 \times 15 \times \pi/180 \approx 1700km$. Plugging those in we get,

$$KE_{max} = \frac{N^2(\Delta y)^2}{8}{\mu_\theta}^2 \approx 40J/kg \tag{1}$$

This is consistent with observations, with implied velocities on the order of $\sqrt{2KE_{max}} = 10m/s$. It is remarkable how such a simple view of the storms seems to work. (NB: This view was pioneered by Prof Eric Eady at Imperial College in the 1950s and it still provides the basic theoretical understanding behind the complicated numerical simulations carried out in climate centers across the world.)

Q7.

(i) Simple harmonic motion at angular frequency N, i.e., $z(t) = A\cos Nt + B\sin Nt$. From the initial conditions $z(0) = 0$ and $dz/dt(0) = w_o$, we get $z(t) = (w_o \sin Nt)/N$. For the values given, the period of the motion is $2\pi/N \approx 10mn$. After $1mn$, it reaches a height of $5.6m$ above its initial position (and is thus just in the initial ascending part of its oscillatory trajectory).

(ii) The motion is unstable and of the form $z(t) = Ae^{Nt} + Be^{-Nt}$. Using the initial conditions, we get $z(t) = w_o(e^{Nt} - e^{-Nt})/2N$. This provides $z = 6.4m$ after $1mn$.

(iii) In this case the parcel's equation predicts $dw_p/dt = 0$, and hence that the parcel will keep rising with the same vertical momentum (the buoyancy force is zero and so there is no upward or downward acceleration provided by the environment). After one minute the height reached is $10 \times 10^{-2} \times 60 = 6m$.

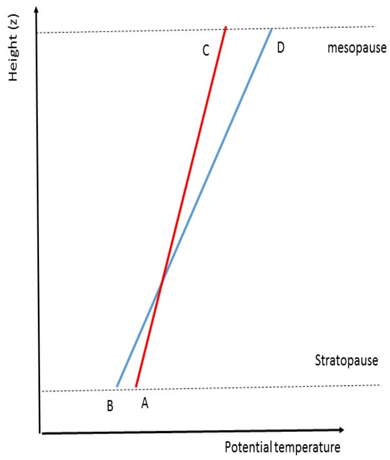

Figure 1: Schematic θ profile for the summer (red) and winter (blue) mesosphere.

Q8.

(i) The winter hemisphere is nearly isothermal so it has a lapse rate approximatively equal to $0K/km$. The summer hemisphere temperature profile varies by about $100K$ between the mesopause and the stratopause, i.e., a lapse-rate $\Gamma \approx 100K/40km = 2.5K/km$.

(ii) Both lapse rates are weaker than the dry adiabatic lapse rate (the one to consider here since there is no significant amount of water vapour in the mesosphere) so the θ profiles are stable and should increase with height. From the Fig. in Chapter 1, the mesopause and the stratopause have roughly the same pressure in the winter and summer hemispheres, which allow to sketch the potential temperature as in Fig. 1 above.

(iii) The Brewer-Dobson circulation requires air to ascend in the summer hemisphere and descend in the winter hemisphere. This is entirely consistent with the air being heated in the summer hemisphere (absorption of solar radiation by ozone and oxygen, see Chapter 2) and so increasing its θ as it moves supward (from A to C). Conversely, the circulation demands descending motion in the winter hemisphere, which is consistent with parcels cooling and decreasing their θ via radiative cooling in the infrared (from D to B).