Solution to Chapter 4's problems

Q1. Calculation of the Rossby number indicates that the system is not in geostrophic balance since $R_o = 10/(10^{-4} \times 10km) = 10 \gg 1$. For the hydrostatic approximation, we must check the magnitude of $\overline{\alpha}\partial P'/\partial z$ compared to the vertical acceleration. Using $\overline{\alpha} \approx 1kgm^{-3}$, one gets $\overline{\alpha}\partial P'/\partial z \approx 10hPa/10km = 0.1ms^{-2}$. Likewise, the vertical acceleration is $w\partial w/\partial z \approx 1/10km = 10^{-4}ms^{-2}$ but it also has the contribution $u\partial w/\partial x \approx 10 \times 1/10km = 10^{-3}ms^{-2}$. So the hydrostatic approximation holds for this system.

Q2. The plane is moving along a latitude circle, eastward, and at constant pressure. Since we're told that the pressure surface $P = 100hPa$ slopes downward along the motion ($\Delta z = 6000 - 5750 = 250m$), it means that there is a pressure difference at fixed height (high to the west, low to the east). From geostrophic balance, this pressure difference must, in the Northern Hemisphere, drive a motion towards the equator hence there will be a drift towards lower latitudes.

To make quantitative statements, we need to make a few additional assumptions. I am going to assume that the east-west pressure gradient is on the order of $10hPa/1000km$ (typical of large scale systems in midlatitudes). At a height of $6km$, the density is roughly the surface value ($\approx 1kg/m^3$) times $e^{-6/8}$ (assuming a scale height of $8km$) so the specific volume is roughly $e^{6/8} = 2m^3/kg$. Hence,

$$v = \frac{\alpha}{f}\frac{\partial P}{\partial x} \approx \frac{2}{2\Omega \sin 45^\circ}\frac{10hPa}{1000km} \approx 20m/s \tag{1}$$

In a time $\Delta t = 1h$, the equatorward drift is $v\Delta t = 72km$, i.e., a drift in latitude slightly less than a degree.

NB: I haven't actually used the value $\Delta z = 250m$. A student (Adrian) suggested this could be included by using the equality used in section 4.2.3 (thermal wind),

$$\alpha \left(\frac{\partial P}{\partial x}\right)_z = \alpha \left(\frac{\partial z}{\partial x}\right)_P \left(\frac{\partial P}{\partial z}\right)_x = -g \left(\frac{\partial z}{\partial x}\right)_P \tag{2}$$

We know Δz so it is just a matter of guessing how far a plane moves in one hour (Δx) to estimate the pressure gradient term as $\alpha\partial P/\partial x \simeq g\Delta z/\Delta x$. A value of $1000km/h$ is not unreasonable, yielding $\Delta x = 1000km$ and a velocity

$v \simeq 24m/s$.

Q3.

(i) The parcel goes around in circles of radius $R\cos\phi$, with angular velocity $\Omega + u/R\cos\phi$. It thus experiences a centrifugal force $(\Omega + u/R\cos\phi)^2 R\cos\phi$. This force has components $\cos\phi\boldsymbol{k} - \sin\phi\boldsymbol{j}$ in the local coordinate system. Hence the particle feels a northward acceleration $-\Omega^2 R\cos\phi\sin\phi - u^2\tan\phi/R - 2\Omega u\sin\phi$. The first term depends only on position and is a contribution to the apparent gravity $\boldsymbol{g}'$. The second and third terms are readily seen in eq. (4.31). Conversely, projecting onto the vertical direction, we obtain an upward acceleration $\Omega^2 R\cos^2\phi + 2\Omega u\cos\phi + u^2/R$. The first term is again contributing to $\boldsymbol{g}'$ while the second is identified with the Coriolis acceleration. The third term is indeed present in (4.27).

(ii) (a) Angular momentum is the product of azimuthal velocity and radius hence the angular momentum of the parcel at rest is $\Omega R^2\cos^2\phi$.

(b) As the parcel moves north it gets closer to the axis of rotation ($R\cos\phi$ decreases). If it conserves its angular momentum, the parcel must thus increase its azimuthal velocity, implying that an eastward acceleration must have taken place.

(c) This is a consequence of the new angular velocity $\Omega + u/R\cos(\phi + \delta\phi)$ and the new radius $R\cos(\phi+\delta\phi)$.

(d) Conservation of angular momentum yields $\Omega R^2\cos^2\phi = [\Omega + u/R\cos(\phi+\delta\phi)][R\cos(\phi+\delta\phi)]^2$. Using $\cos(\phi+\delta\phi) \approx \cos\phi - \delta\phi\sin\phi$, we obtain, to leading order in small quantities $(\delta u, \delta\phi)$, $0 = -\Omega R^2 2\delta\phi\sin\phi\cos\phi + \delta u R\cos\phi$. From this, $\delta u = 2\Omega R\sin\phi\delta\phi$ (sign makes sense since poleward motion $v > 0$ must lead to faster cyclonic azimuthal flow $u > 0$), so that, since $v \equiv R\delta\phi/\delta t$ (in the limit of very small $\delta\phi, \delta t$), $\delta u/\delta t = 2\Omega\sin\phi v$.

(iii) If the parcel is initially accelerated upwards, it will gain a height δz in a time δt. Note that it will still increase its latitude by $\delta\phi$ (as in (ii)b, a momentum impulse is given to the parcel in the northward direction) so the calculation is the same as before, but with a slightly different version of the conservation of angular momentum: $\Omega(R+\delta z)^2\cos^2\phi = [\Omega + u/(R+\delta z)\cos(\phi+\delta\phi)][(R+\delta z)\cos(\phi+\delta\phi)]^2$. Noting that $w = \delta z/\delta t$ we get, $\delta u/\delta t = 2\Omega v\sin\phi - 2\Omega w\cos\phi$.

Q4.

(i) Start by rewriting (4.42) as,

$$u_2 - u_1 = \frac{g}{f}\frac{\partial}{\partial y}(z_1 - z_2) \tag{3}$$

Then, approximate $z_1 - z_2$ as,

$$z_1 - z_2 = \frac{R_d}{g}\left(\ln \frac{P_2}{P_1}\right)\overline{T} \tag{4}$$

The result follows.

(ii) Applying the formula with $f = 2\Omega \sin 45°$ yields $u_1 - u_2 \approx 90m/s$. Assuming the winds to be small near the Earth's surface, we get $u_1 \approx 90m/s$. This is a very strong jet! (most likely due the deep extent assumed $1000 - 200hPa$).

Q5⋆. The geometry of the problem is summarized in Fig. 1a.

(i) We need to scale the relative magnitude of ζ and f. Typically, $\zeta = U/L$ so the ration $\zeta/f \approx U/(fL)$. In midlatitudes, $f \approx 10^{-4}s^{-1}$ so, using $U = 2cm/s$ and $L = 4000km$ (rough size of the Atlantic basin), we obtain $\zeta/f \approx 5.10^{-5}$. We can thus indeed safely neglect the relative vorticity (you may notice that this question readily provides another interpretation for the Rossby number, namely the ratio of the relative vorticity to the planetary vorticity in the flow). If the gyre occupies the range of latitude between $10°N$ and $50°N$, the loss of vorticity in the equatorward leg of the gyre will be $\Delta\zeta_a \approx f(10°N) - f(50°N) = -8.6.10^{-5}s^{-1}$. If you're interested to know more about this, the loss of absolute vorticity arises because the net wind effect is to squeeeze ocean columns (a bit like a ballerina wrapping onto herself spins less fast).

(ii) Repeating the calculation for the poleward leg, we get $\zeta/f \approx 2ms^{-1}/(30\times km \times 10^{-4}s^{-1}) = O(1)$. So we cannot neglect the relative vorticity of the flow in the Gulf Stream.

(iii) To complete a loop, and so come back to their initial state, water parcels must regain in the Gulf Stream the loss of ζ_a they experienced during the equatorward journey. Thus $\zeta + f$ must increase in the poleward leg.

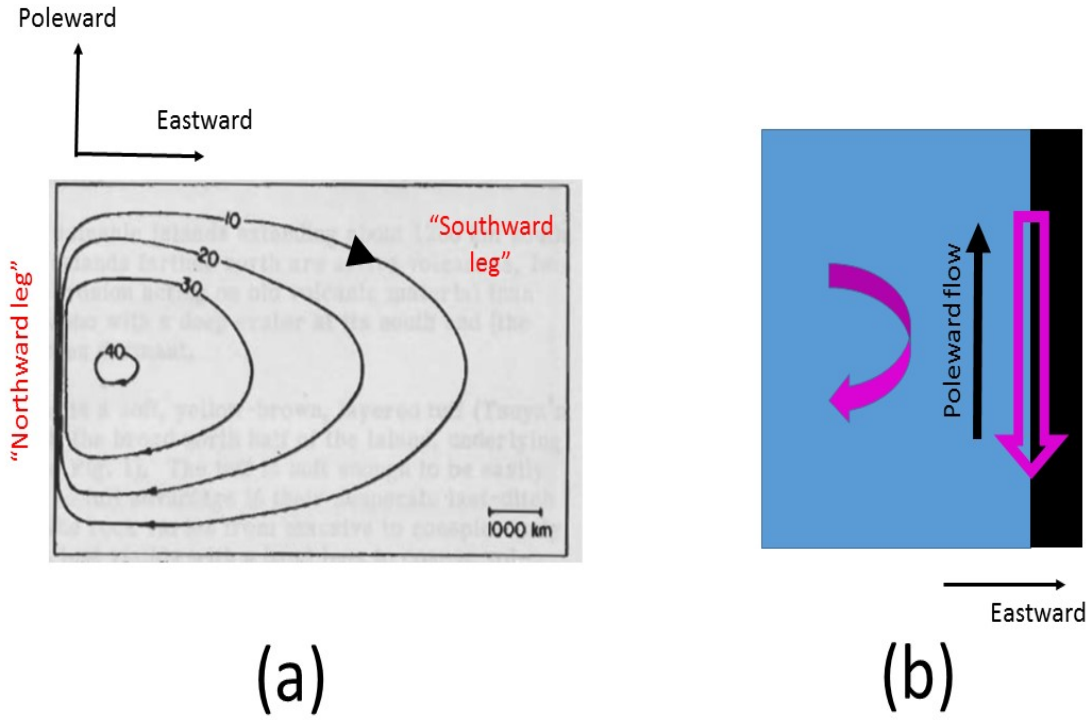

Figure 1: (a) Schematic of a subtropical gyre in the longitude/latitude plane. (b) A failed attempt to create a Gulf Stream off the coast of Portugal! This schematic zooms in on a portion of the subtropical gyre near the coast of Portugal. An hypothetical poleward flow there would be associated with an equatorward frictional force (magenta arrow) greater near the coast. This would tend to make the fluid spin in a clockwise way (curly magenta arrow).

(iv) The idea here is that friction must provide the mechanism increasing the absolute vorticity of the fluid parcels. You need to do a drawing to see the difference between each side of the Atlantic (Fig. 1b). If the Gulf Stream were on the west coast of Portugal, the frictional force acting on the poleward flow would tend to rotate the flow in a clockwise way. This is a gain of negative absolute vorticity (like in an anticyclone), so cannot provide the required gain. If you plot the same diagram for the east coast of the US, you'll find that this time frictional effects act in the right direction.

NB: The first mathematical model to explain the presence of the Gulf Stream was put forward in 1948 in a paper by Henry Stommel entitled "The western intensification of wind-driven ocean currents". Stommel was one of the founding fathers of physical oceanography.

Q6. The analogy with low pressure systems is completely inadequate. For one thing, the Rossby number for this bath tube flow is very large (taking $U = 3cm/s$, $L = 10cm$ I get $R_o \approx 2000$) so the geostrophic balance on which the different sense of circulation is based is not relevant to the bath tube problem. Another way to see this is that a bath tube flow has no vorticity (remember the video shown in the Lecture on vorticity), while low pressure systems are full of it.

The analogy with low pressure system is not correct but, as the movie on vorticity (see Blackboard) shows, if care is taken in a lab, one can visualize the effect of the Earth's rotation in a sink flow. If the movie apparatus is used in the Northern or Southern hemispheres (i.e., with a large tank and a very slow drain), the vorticity-meter will rotate counter-clockwise in the North but clockwise in the South. In a real sink flow, you will more likely notice the gyral motion associated with the angular momentum of the flow (which has no vorticity and thus would not make the vorticity-meter turn). This gyral motion can be clockwise or anti-clockwise depending on the initial conditions.

Q7.

(i) Vorticity is a vector defined as the curl of the velocity field. Its vertical component is particularly important as the geostrophic flow makes an important contribution to it. It is important to understand and predict the weather because, unlike the geostrophic approximation which is purely diagnostic, the vorticity equation predicts evolution through time.

(ii) Start with conservation of the vertical component of vorticity ($\zeta_a = f + \zeta$) for a 2D (horizontal) flow,

$$\frac{D(f+\zeta)}{Dt} = 0 \tag{5}$$

where $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$. We are interested here in the linearised version of this equation, with $u = U + u'$ and $v = 0 + v'$ (no background flow in the North South direction),

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\frac{df}{dy} = 0 \tag{6}$$

where $\zeta' = \partial v'/\partial x - \partial u'/\partial y$. In addition, we assume no y dependence for U, u' and v', hence,

$$[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}]\frac{\partial v'}{\partial x} + \beta v' = 0 \tag{7}$$

This starts to look like the answer. To get there, consider solutions of the form $v' = e^{ik(x-ct)}$, leading to $\partial v'/\partial t = -c\partial v'/\partial x$. Hence the formula given in the exam.

To obtain the value of c, expand the second derivative as a function of the zonal wavenumber k,

$$(U-c)(-k^2)v' + \beta v' = 0 \tag{8}$$

This leads to $c = U - \beta/k^2$.

(iii) For a stationary wave $c = 0$, leading to $k = \sqrt{\beta/U}$. At $50°N$, $\beta = 2\Omega\cos(50°)/R = 1.46 \times 10^{-11} m^{-1}s^{-1}$, leading, for $U = 50ms^{-1}$ to $k = 2\pi/(11,590km)$. The parameter β would be larger at a lower latitude, so the curve $c(k)$ would shift towards the right: this wave would appear to propagate westward ($c < 0$).