Notes on symmetric instability / “slantwise convection”

A. Czaja, August 2011.

1 The problem

The simplest model for symmetric instability is provided by considering an unbounded domain filled with a fluid under rotation and in thermal wind balance with an horizontal temperature gradient. In terms of geometry, take the background thermal wind $v_o(z)$ to be in the y direction and assume $\partial/\partial y = 0$ everywhere (2D symmetry). Then:

$$f_o \frac{\partial v_o}{\partial z} = \frac{g}{\theta_o} \frac{\partial \theta_o}{\partial x} \tag{1}$$

in which θ_o is the background temperature gradient (f-plane and Boussinesq approximation are used). This background flow has a vorticity vector $\boldsymbol{\zeta}_o$ whose components in the (x, y, z) plane are:

$$\boldsymbol{\zeta}_o = (-v_{o,z}, 0, f_o) \tag{2}$$

The key question is whether such vortex can be destabilized by small perturbations.

2 The solution

In an unbounded domain, the answer is that the vortex will be destabilized by a very simple form of motion: ascent and descent along constant temperature surfaces θ (Bennets and Hoskins, 1979). To understand why this happens, write the momentum balance for a small perturbation (u', v', w'):

$$\begin{aligned} \frac{\partial u'}{\partial t} - f_o v' &= -\frac{1}{\rho_o}\frac{\partial P'}{\partial x} && (3) \\ \frac{\partial v'}{\partial t} + w' v_{o,z} + f_o u' &= -\frac{1}{\rho_o}\frac{\partial P'}{\partial y} = 0 && (4) \end{aligned}$$

These can be further simplified for motions along θ-surface because these, in the limit of the hydrostatic approximation, do not create temperature perturbations and thus no pressure perturbations. The $\partial P'/\partial x$ term can then be dropped and the momentum equations can be simply rewritten as:

$$\frac{\partial u'}{\partial t} - f_o v' = 0 \quad (5)$$

$$\frac{\partial v'}{\partial t} + w' v_{o,z} + f_o u' = 0 \quad (6)$$

In absence of thermal wind shear ($v_{o,z}$), i.e., when $\boldsymbol{\zeta}_o$ is purely vertical, the vortex is stable since all that can happen are "inertial oscillations" whose energy can not change because the Coriolis force does not do work on the flow:

$$\frac{\partial u'}{\partial t} - f_o v' = 0 \quad (7)$$

$$\frac{\partial v'}{\partial t} + f_o u' = 0 \quad (8)$$

$$\frac{\partial}{\partial t}(u'^2 + v'^2) = 0 \quad (9)$$

With thermal wind however, the vertical advection term $w' v_{o,z}$ (the "inertial acceleration") opposes the stabilizing effect of the $f_o u'$ term since, along a θ surface these are anti-correlated (Fig. 1). Whether there is stability or not thus depends on the relative magnitude of the $w' v_{o,z}$ and $f_o u'$ terms. It would be ideal to be able to write their sum as a single term –this can be done by introducing $\boldsymbol{\zeta}_o$.

To get there, first use the fact that along a θ surface, one has:

$$u'\theta_{o,x} + w'\theta_{o,z} = 0 \quad (10)$$

As a result,

$$w' v_{o,z} + f_o u' = \frac{u'}{\theta_{o,z}}(f_o \theta_{o,z} - v_{o,z}\theta_{o,x}) \quad (11)$$

From the definition (2), this is nothing else than:

$$w' v_{o,z} + f_o u' = \frac{u'}{\theta_{o,z}} \boldsymbol{\zeta}_o . \nabla \theta_o \quad (12)$$

Defining ζ_θ as the projection of $\boldsymbol{\zeta}_o$ along the normal to a θ surface, i.e.,

$$\zeta_\theta \equiv \frac{\boldsymbol{\zeta}_o . \nabla \theta_o}{\theta_{o,z}} \quad (13)$$

the momentum equations can finally be rewritten as:

$$\frac{\partial u'}{\partial t} - f_o v' = 0 \qquad (14)$$

$$\frac{\partial v'}{\partial t} + \zeta_\theta u' = 0 \qquad (15)$$

The analogy with the inertial oscillation is clear and growth can occur if $f_o \zeta_\theta < 0$. Geometrically, this simply means that the vortex is destabilized when $\boldsymbol{\zeta}_o$ is more horizontal than the θ_o surfaces (Fig. 1). Conversely, the vortex is neutral to the perturbations when $\boldsymbol{\zeta}_o$ is aligned with the θ_o surfaces. It is stable to the perturbations when more vertical than the θ_o surfaces. A convenient alternative to $\boldsymbol{\zeta}_o$ is to introduce the scalar M (called "absolute momentum" by Eliassen (1962) in analogy with absolute vorticity being the sum of planetary and relative components) defined by:

$$M \equiv f_o x + v \qquad (16)$$

It is readily seen that $\boldsymbol{\zeta}_o$ is aligned with surfaces of constant $M_o = f_o x + v_o$ since $\boldsymbol{\zeta}_o.\nabla M_o = -v_{o,z} f_o + f_o v_{o,z} = 0$ –this is why surfaces of constant M_o are shown in Fig. 1.

3 Useful properties of the solution

The ascent / descent along isentropes illustrated in the previous solution comes up systematically in numerical simulations of SI, even in the presence of boundaries (Miller, 1984; Thorpe and Rotunno, 1989; Thomas and Taylor, 2010). There are a couple of useful properties of this solution that need to be highlighted.

In terms of horizontal scale, the perturbation in Fig. 1 will adopt a wavelength set by the depth of the unstable zone and the slope of the isentropes (Emanuel, 1979).

In terms of energy, the perturbation feeds on the kinetic energy of the thermal wind flow. Indeed, from the momentum equations, one readily gets that:

$$\frac{1}{2}\frac{\partial}{\partial t}(u'^2 + v'^2) = -w'v'v_{o,z} \qquad (17)$$

A growing perturbation tends to reduce the thermal wind shear ($w'v' < 0$) by transporting meridional momentum downward. At first sight this looks like the perturbation is trying to neutralize the background flow but, in an infinite domain, this will not happen (the PV is negative everywhere initially and will not become zero by pure advection since fluid parcels conserve their

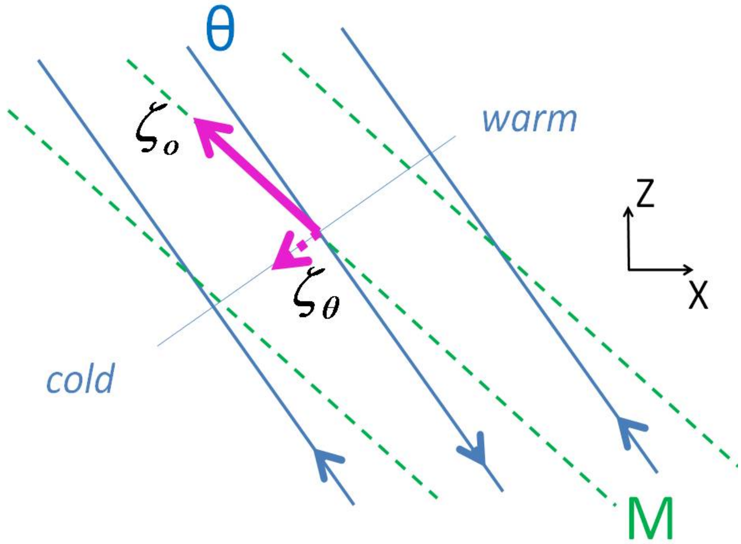

Figure 1: The most unstable perturbation for the background thermal wind flow is a simple ascent / descent along isentropes. The projection of the absolute vorticity vector onto the isentropic gradient is negative and this means that inertial acceleration dominates over the Coriolis force in the y-momentum equation.

PV –this is a very striking feature of SI emphasized by Thorpe and Rotunno, 1989). The finite amplitude evolution is thus that the M surfaces become more and more vertical (reduced vertical windshear) and that the θ surfaces follow in order to conserve PV.

Because of $v'w' < 0$, there will be, at fixed height, an alternation of northward/southward flow. Another way to say this is that the background frontal wind breaks up into narrower jets.

The perturbation also transports zonal momentum downwards since, from (10), $u'w' \propto -u'^2\theta_{o,x}/\theta_{o,z} < 0$. Thus one anticipates that, in addition to reducing the thermal wind shear, a "coldward" jet will be produced above a localized region experiencing SI and a "warmward" jet below (jets in the plane orthogonal to the background thermal wind). This is schematized in Fig. 2 (strictly speaking this really applies to an unstable zone localized in z but infinite in the horizontal direction since otherwise $v_{o,x}$ could change the dynamics).

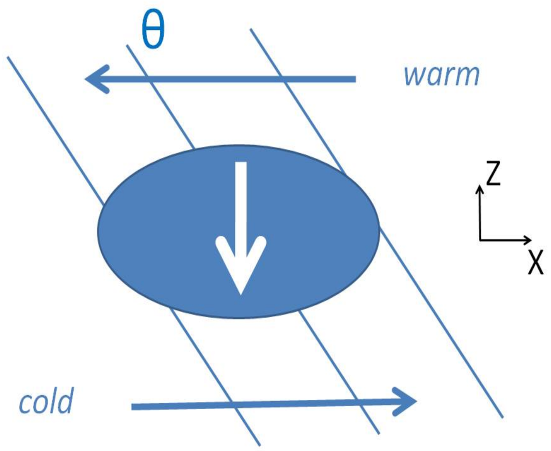

Figure 2: The white arrow represents the transport of zonal and meridional momentum by a localized (shaded ellipse), growing, perturbation. The associated impact is to reduce the background meridional wind shear and to produce jets in the plane orthogonal to the latter (filled arrows). This transverse circulation will tend to reduce the horizontal temperature gradient but enhance the vertical temperature gradient.

An interesting aspect of Fig. 2 is that even though the solution illustrated in Fig. 1 does not transport heat ($\theta' = 0$), the large scale circulation induced by the growing perturbation will tend to tilt the isentropes towards the horizontal, i.e., transport heat upwards.