Notes for the "Oceans in Weather and Climate" lecture – "Ocean-atmosphere coupling: theory and observations"

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March 13, 2018

Contents

1 The Hoskins-Karoly model

The focus of this section is on section 3a of Hoskins and Karoly (1981, hereafter HK81), aiming at deriving their eqs. (3.1) and (3.2) from "first principles" and at explaining some of the meteorological jargon they used. The same notations are used as in HK81.

1.1 The physical model: the background state

We are looking at linear perturbations developing on the Jet Stream, perturbations caused by a prescribed heating and cooling within the atmosphere. This heating/cooling is somehow related to the ocean circulation but we will not attempt at doing so (prescribed heating/cooling).

The Jet Stream is simply represented as a background eastward flow $\overline{u}$ which is linearly increasing with height z:

$$\overline{u}(z) = \overline{u}_z z \tag{1}$$

In this equation $\overline{u}_z$ is the constant vertical derivative of $\overline{u}$. It is important to emphasize that throughout the paper we are talking about motions relative to an observer moving with the Earth. So the total zonal (=west to east) wind is actually $\overline{u} + \Omega R$ in which Ω is the Earth's angular rotation and R its radius. In other words air parcels in the Jet Stream rotate faster than the

Earth. Also, from now on, all variables with an overbar refer to values for the background state.

A Coriolis force acts on eastward moving parcels, accelerating them towards the equator. If the background state is a state of equilibrium something must oppose the Coriolis force: this is the pressure gradient force. To produce a poleward acceleration we must have high pressure at low latitudes and low pressure at high latitudes. This simplification of the momentum equation is called the geostrophic balance:

$$f\overline{u} = -\frac{1}{\rho_o}\frac{\partial \overline{P}}{\partial y} \tag{2}$$

In the latter, y denotes the north-south direction (in meters, i.e., $dy = Rd\phi$ in which R is ϕ is the latitude in radians), P is the pressure and f is the Coriolis parameter:

$$f \equiv 2\Omega \sin\phi \tag{3}$$

Note the constant term ρ_o in (2), which is a reference density (this does not have to be taken constant for geostrophic balance but it will simplify the subsequent derivation).

Equation (2) is only the component of Newton's law in the meridional (=north-south) direction. In the vertical, we assume that gravity is opposed by pressure gradients, i.e., high pressure at the Earth's surface and low pressure aloft. This the hydrostatic approximation:

$$\overline{\rho} g = -\frac{\partial \overline{P}}{\partial z} \tag{4}$$

The trick of keeping the full ρ on the left hand side of (4) but the reference value in (2) is called the Boussinesq approximation (it does not change the basic physics in HK81 but simplifies greatly the maths).

There is no need to consider the zonal component of Newton's law of motion because there is no meridional motion and no pressure variations along a latitude circle in the background state, i.e.:

$$\overline{v} = 0 \quad \text{and} \ \frac{\partial \overline{P}}{\partial x} = 0 \tag{5}$$

So far we have only talked about Newton's law. To know the pressure field we need to make a few more assumptions about the thermodynamics of the model. In fact it is very simple, we assume the following equation of state:

$$\rho = \rho_o(1 - \alpha\theta) \tag{6}$$

in which α is a constant and θ is "potential temperature" (this is meteorologists' entropy: $s = c_p \log \theta$).

In summary, the background atmospheric state is a purely zonal flow in geostrophic and hydrostatic balance with the pressure field. If we do not perturb it, nothing happens. If we heat or cool the atmosphere, the entropy will increase or decrease, density perturbations will be created through (6) and this will in turn cause pressure perturbations through (4). The problem addressed by HK81 is to characterize the steady perturbations developing in response to an imposed heating or cooling: is a low pressure created in response to heating? does this affect the whole air column or just the surface? etc.

1.2 The physical model: perturbations

We assume that the forcing is small enough that perturbations (denoted by a prime) will be small compared to their values in the background state. For example, $u = \overline{u} + u'$ with $u' \ll \overline{u}$ or $\rho = \overline{\rho} + \rho'$ with $\rho' \ll \overline{\rho}$. For some variables like the north-south wind v however this cannot be since $\overline{v} = 0$ and thus $v = v'$. Perturbations are assumed independent of time but 3D, i.e. $u' = u'(x, y, z), v' = v'(x, y, z)$ etc.

To get to eq. (3.1) we need to write the two horizontal components of Newton's law for small perturbations:

$$\overline{u}\frac{\partial u'}{\partial x} - fv' = -\frac{1}{\rho_o}\frac{\partial P'}{\partial x} \quad (7)$$

$$\overline{u}\frac{\partial v'}{\partial x} + fu' = -\frac{1}{\rho_o}\frac{\partial P'}{\partial y} \quad (8)$$

In the vertical we still assume that perturbations follow the hydrostatic balance:

$$\rho' g = -\frac{\partial P'}{\partial z} \tag{9}$$

or, using (6):

$$-\alpha\rho_o\theta' g = -\frac{\partial P'}{\partial z} \tag{10}$$

Now take the x-derivative of (8) and substract the y-derivative of (7) and simplifies the notations, as in HK81, by writing for example $\partial u'/\partial x \equiv u'_x$. We obtain:

$$\overline{u}(v'_x - u'_y)_x + (fu')_x + (fv')_y = 0 \tag{11}$$

since the pressure terms cancel and $\overline{u}$ is independent of x, y. We are almost at eq. (3.1) in HK81! To finish the job, we must say something about mass conservation. This is where again we use the Boussinesq trick and replace mass conservation by volume conservation:

$$u_x + v_y + w_z = 0 \tag{12}$$

In (12), w is the vertical velocity. Applied to the background state this equation just says that $\overline{w}_z = 0$. With $\overline{w} = 0$ at the ground (no mountains...), it produces $\overline{w} = 0$ everywhere. For the perturbations, it becomes:

$$u'_x + v'_y + w'_z = 0 \tag{13}$$

Introducing the latter in (11)

$$\overline{u}(v'_x - u'_y)_x + u' f_x + v f_y = f w'_z \tag{14}$$

From the definition (3), the Coriolis parameter is only a function of latitude, i.e. $f_x = 0$. Defining β as

$$\beta \equiv f_y = 2\Omega \cos\phi / R \tag{15}$$

we obtain

$$\overline{u}(v'_x - u'_y)_x + \beta v' = f w'_z \tag{16}$$

This is eq. (3.1) of HK81 after introducing the relative vorticity ζ:

$$\zeta \equiv v_x - u_y \tag{17}$$

This quantity is a measure of spin of air parcels around the vertical axis.

To get to HK81's eq. (3.2) is more straightforward because it is simply a hidden version of the 2nd law of thermodynamics: $ds/dt = \dot{Q}/T$ in which $\dot{Q}$ is the heating rate and T temperature. The d/dt here reflects the change following an air parcel to which we apply the 2nd law, i.e.

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{18}$$

To see where this comes from, simply think $s = s(x, y, z, t)$ and apply the usual partial differentials of thermodynamics:

$$ds = (\frac{\partial s}{\partial x})dx + (\frac{\partial s}{\partial y})dy + (\frac{\partial s}{\partial z})dz + (\frac{\partial s}{\partial t})dt \tag{19}$$

The connection should now be clear, acknowledging that $u \equiv dx/dt$, etc.

Linearizing the 2nd law for a steady state ($\partial/\partial t = 0$) we get:

$$\overline{u}s'_x + v'\overline{s}_y + w'\overline{s}_z = (\frac{\dot{Q}}{T})' = \frac{\dot{Q}'}{\overline{T}} \tag{20}$$

Note the background gradients $\overline{s}_y$ and $\overline{s}_z$ which are needed to sustain the pressure gradients in (2) and (4). Using the definition of θ, this can be rewritten as:

$$\overline{u}\theta'_x + v'\overline{\theta}_y + w'\overline{\theta}_z = \frac{\overline{\theta}}{c_p\overline{T}}\dot{Q}' \tag{21}$$

This is eq. (3.2a) of HK81 in a disguised form. The version actually needed is eq. (3.2b) which we can get to after a few more manipulations. To see this derive first a "thermal wind" relationship for the background state, by combining (2) and (4):

$$f\overline{u}_z = -g\alpha\overline{\theta}_y \tag{22}$$

Likewise for the perturbations we make the geostrophic approximation[1] to (7):

$$-fv' = -\frac{1}{\rho_o}\frac{\partial P'}{\partial x} \tag{23}$$

which, combined with (10), provides a thermal wind relation for the pertur-

[1] Anticipating on section 3, you can scale the full equation and you will see that the geostrophic approximation is valid as long as the non dimensional number $\overline{u}/fL$ is small compared to unity. This implies that we restrict ourselves to horizontal scales $L \gg \overline{u}/f \simeq 10/10^{-4} = 100km$.

bations:

$$f v'_z = g\alpha\theta'_x \tag{24}$$

This allows to rewrite (21) as:

$$(f/g\alpha)\overline{u}v'_z - (f/g\alpha)v'\overline{u}_z + w'\overline{\theta}_z = \frac{\overline{\theta}}{c_p\overline{T}}\dot{Q}' \tag{25}$$

Multiply by $g\alpha$, this becomes:

$$f\overline{u}v'_z - fv'\overline{u}_z + w'g\alpha\overline{\theta}_z = \frac{g\alpha\overline{\theta}}{c_p\overline{T}}\dot{Q}' \tag{26}$$

Introduce $Q \equiv \frac{g\alpha\overline{\theta}}{c_p\overline{T}}\dot{Q}'$ and

$$N^2 \equiv g\alpha\overline{\theta}_z \tag{27}$$

and we finally obtain eq. (3.2b) of HK81:

$$f\overline{u}v'_z - fv'\overline{u}_z + w'N^2 = Q \tag{28}$$

Note that N is a well known quantity: the buoyancy frequency of the atmosphere. It is the frequency at which air parcels oscillate vertically due to restoring effect of buoyancy forces. This frequency is quite rapid, $N \simeq 10^{-2}s^{-1} = 1/100s$.

1.3 The predicted response of the atmosphere to heating and cooling

Now comes the interesting part of making predictions regarding the perturbations caused by Q without solving the equations! To do this we perform a scaling analysis of the equations, that is, we look which terms are more important for a given horizontal scale L (the scale of the prescribed heating considered). We do not distinguish between zonal (L_x) and meridional (L_y) scales, i.e. we take $L \simeq L_x \simeq L_y$. In addition we assume a similar velocity scale for u' and v' and we simply denote it by v'.

To see how it works, let's focus on the vorticity equation:

$$\overline{u}\zeta'_x + \beta v' = f w'_z \tag{29}$$

The first term on the l.h.s is a local increase of spin when the flow brings high vorticity fluid. This scales as $\overline{u}v'/L^2$ using the definition (17). The second term is the local increase in spin when the flow brings fluid from the north. This occurs because the projection of the Earth's rotation onto the local vertical is optimal at the North pole and zero at the equator (the "β-effect"). This second term scales simply as $\beta v'$ so that the ratio of the first to the second term is:

$$\overline{u}v'/L^2 \div \beta v' = \frac{\overline{u}}{\beta L^2} \tag{30}$$

This shows that there is a typical horizontal lengthscale $(\overline{u}/\beta)^{1/2}$ in this problem and that perturbations with $L \gg (\overline{u}/\beta)^{1/2}$ have a simple vorticity balance:

$$\beta v' \approx f w'_z \quad \text{for } L \gg (\overline{u}/\beta)^{1/2} \tag{31}$$

To see how large is this reference scale, we note that in midlatitudes $\beta \simeq$

$2.10^{-11} m^{-1} s^{-1}$ and that $\overline{u} \simeq 10 m s^{-1}$. This provides:

$$(\overline{u}/\beta)^{1/2} \simeq 700 km \tag{32}$$

So for heating/cooling on a scale of a few thousand km, we can safely use (31). This is a nice simplification of the problem which we will use, focusing from now on on horizontal scales larger than $(\overline{u}/\beta)^{1/2}$. (Alternatively, one can think of the Sverdrup balance as more relevant to low levels where $\overline{u}$ is small).

Introduce now two vertical scales:

$$H_u \equiv \overline{u}/\overline{u}_z \quad \text{and } H_Q \equiv Q/Q_z \tag{33}$$

The Jet Stream goes typically from near zero at the surface to several tens of ms^{-1} at a height of $10km$ in midlatitudes, so $H_u \simeq 10/(30/10km) = 3km$. We'll fix this and will simply consider the case of "deep heating" for which $H_Q \gg H_u$ because it is the case most relevant to ocean-atmosphere interactions in the Tropics and over western boundary currents in midlatitudes.

Consider now the thermodynamic equation (28). It is made of three terms on the l.h.s which are, respectively, advection of entropy in the east-west direction, in the north-south direction, and in the vertical direction. If the first of these dominate, it must be associated with a meridional velocity scale

$$v' \simeq Q H_Q / f \overline{u} \quad \text{(if zonal advection dominates)} \tag{34}$$

Likewise if the 2nd term dominates:

$$v' \simeq Q/f\overline{u}_z = QH_u/f\overline{u} \quad \text{(if meridional advection dominates)} \tag{35}$$

Finally, if the 3rd term dominates (using the simplified vorticity balance):

$$v' \simeq fw'/\beta H_Q = fQ/\beta N^2 H_Q \quad \text{(if vertical advection dominates)} \tag{36}$$

The key assumption is now that the mechanism with smallest v' will dominate (this is a thermodynamic efficiency argument: converting heating/cooling into motion is difficult so it's likely that Nature will opt for the "smallest effort required"). The assumption of deep heating removes the zonal advection so there is only to compare meridional and vertical advection. The ratio γ of the 3rd to the 2nd term is:

$$\gamma \equiv \frac{f^2\overline{u}}{\beta N^2 H_Q H_u} \tag{37}$$

In other words,

$$v'\overline{\theta}_y \approx \frac{\overline{\theta}}{c_p\overline{T}}\dot{Q}' \quad \text{for } \gamma \gg 1 \tag{38}$$

and

$$w'\overline{\theta}_z \approx \frac{\overline{\theta}}{c_p\overline{T}}\dot{Q}' \quad \text{for } \gamma \ll 1 \tag{39}$$

These two limits correspond to very different physical situations. In the case where $\gamma \gg 1$, heating is opposed by cooling due to the advection of cold air from the pole ($v' < 0$). From the vorticity equation (31) this implies downward motion at mid-levels since $w' = 0$ at the surface. From geostrophy, a low pressure must be found to the east of the heating. This is the situation depicted in HK81's Fig. 2b. However, in the case where $\gamma \ll 1$, heating is opposed by cooling due to expansion of a rising air parcel. From the vorticity equation (31) this implies poleward motion ($v' > 0$ in the Northern Hemisphere where $f > 0$) at low-levels since $w' = 0$ at the surface. This implies a low pressure to the west of the heating by geostrophy. This is the situation depicted in HK81's Fig. 2a.

Interestingly, there is a strong dependence of γ with latitude as a result of the factor f^2/β in (37) –see Fig. 1. As a result, the case $\gamma \ll 1$ can be associated with low latitudes and $\gamma \gg 1$ with high latitudes.

1.4 Further comments

The Sverdrup balance and ascent over a heat source is clearly relevant at low latitudes because of the large horizontal scale of the diabatic heating in the Tropics (see more on the interpretation of what the heat source really represents in the next section). In midlatitudes, and in agreement with observations over the Gulf Stream by Minobe (2008), vertical ascent over the heat source is still relevant, as is advection of cold air from the poles to balance the heat source, because Sverdrup balance can be broken (i.e., the

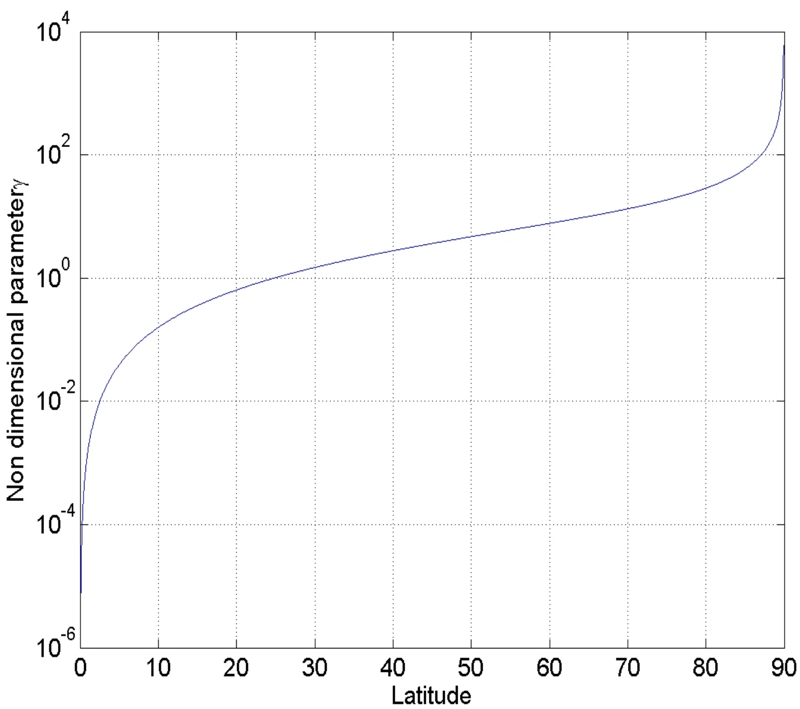

Figure 1: The parameter γ as a function of latitude assuming fixed values $H_Q = 6km, H_u = 3km, N = 10^{-2}s^{-1}$ and $\overline{u} = 10ms^{-1}$.

zonal advection of vorticity cannot be neglected). This is more so for deep heating than shallow heating because $\overline{u}$ is nearly zero near the ground where the heat source is intensified in the latter case.

It is also interesting to realise that even a shallow heating can lead to upper level perturbations, which is particularly vivid in the experiments by Hendon and Hartmann (1982). This is because one needs to reverse the sign of $\partial w'/\partial z$ to bring w' to zero at the tropopause, as required to satisfy the boundary condition ($w' = 0$ at the tropopause). And, from the conservation of vorticity, this requires either $\overline{u}\zeta'_x$ or $\beta v'$ to be non zero at upper levels.

2 What does the heat source really represent?

The perturbations studied in the previous section typically represent climate anomalies, for example, an El Nino situation or an anomalous NAO winter. So the primes are departure from a time mean (e.g., winter time mean). This means that there is in fact an additional term on the r.h.s of the heat conservation equation. If we denote the averaging by a <> and the departures from it by a *, conservation of heat for the averaged variable reads, in general:

$$\frac{\partial < \theta >}{\partial t} + < \boldsymbol{v} > . \boldsymbol{\nabla} < \theta > = < Q_{diab} > - \boldsymbol{\nabla} . < \boldsymbol{v}^\star \theta^\star > \qquad (40)$$

where

$$< Q_{diab} >=< Q_{rad} > + < Q_{phase} > + < Q_{sen} > \tag{41}$$

is the total diabatic heating (the sum of heating due to radiative processes, phase change and temperature difference between air and water). So the Q in HK81 is an apparent heating $< Q_{app} >$ defined as:

$$< Q_{app} >=< Q_{diab} > -\boldsymbol{\nabla}. < \boldsymbol{v}^{\star}\theta^{\star} > \tag{42}$$

This can be quite different from any a priori assumed diabatic heating $< Q_{diab} >$ both at tropical (effect of convection) and extra-tropical (synoptic weather systems) latitudes.