Analogues of Potential Vorticity in Magnetohydrodynamic Fluids

by

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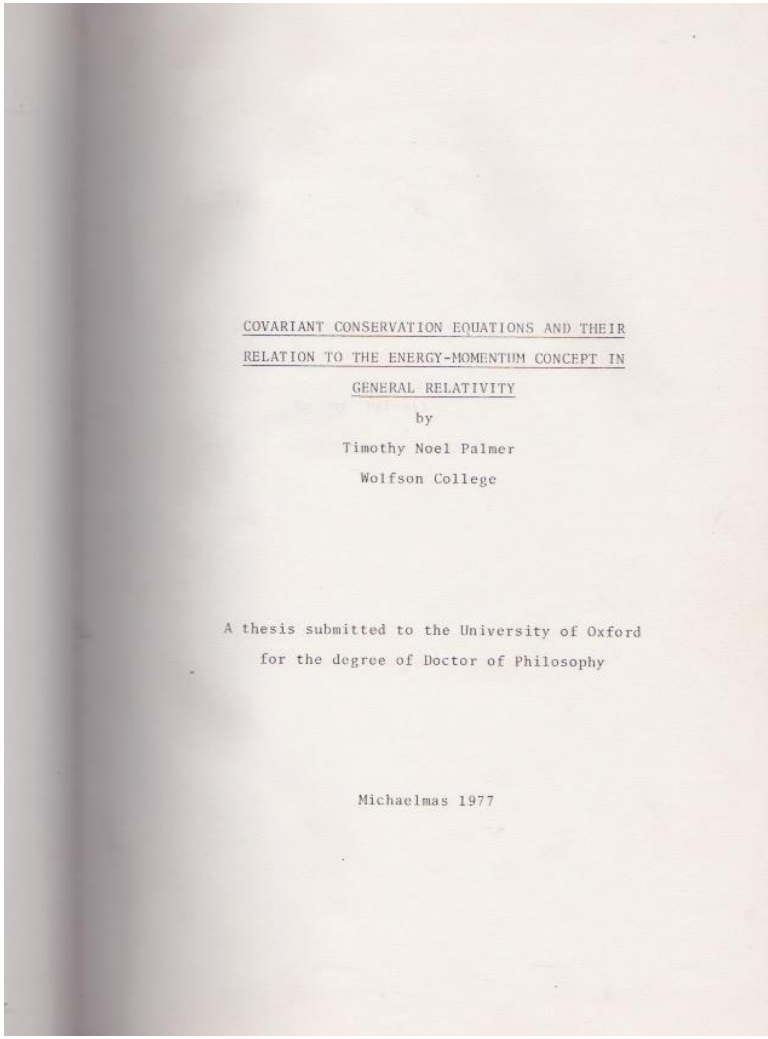

COVARIANT CONSERVATION EQUATIONS AND THEIR
RELATION TO THE ENERGY-MOMENTUM CONCEPT IN
GENERAL RELATIVITY

by

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Wolfson College

A thesis submitted to the University of Oxford
for the degree of Doctor of Philosophy

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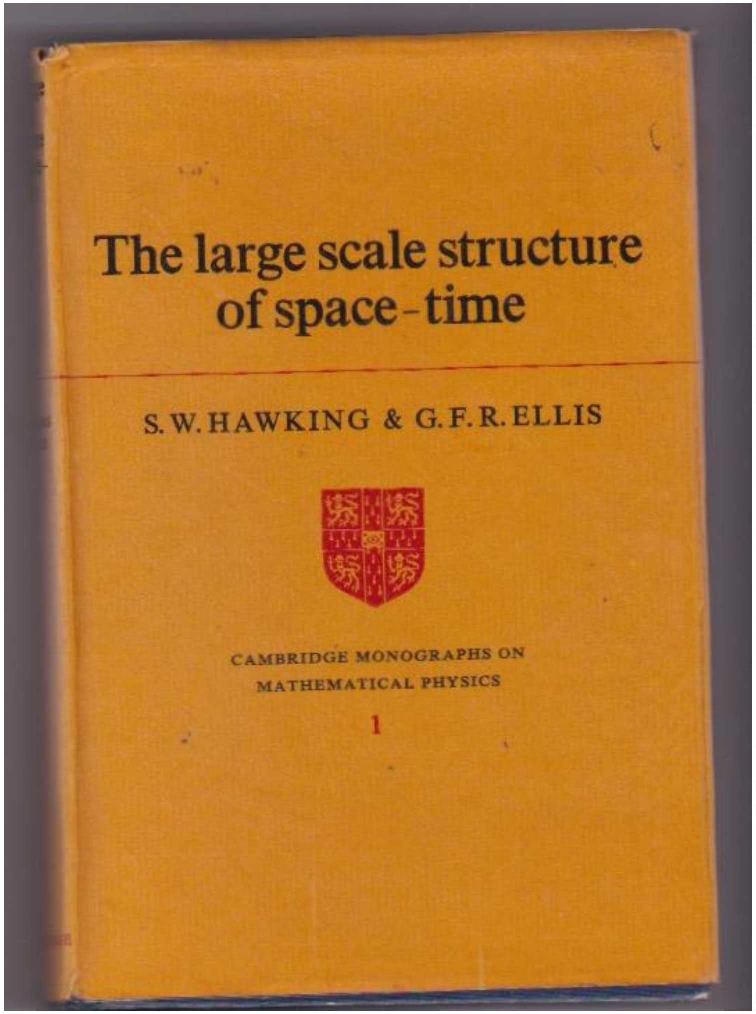


The large scale structure
of space-time

S. W. HAWKING & G. F. R. ELLIS

CAMBRIDGE MONOGRAPHS ON
MATHEMATICAL PHYSICS

1

this case, $n = n'$, and ϕ is both injective and surjective if $r \geqslant 1$; conversely, the implicit function theorem shows that if ϕ_* is both injective and surjective at p, then there is an open neighbourhood $\mathscr{U}$ of p such that $\phi: \mathscr{U} \to \phi(\mathscr{U})$ is a diffeomorphism. Thus ϕ is a local diffeomorphism near p if ϕ_* is an isomorphism from T_p to $T_{\phi(p)}$.

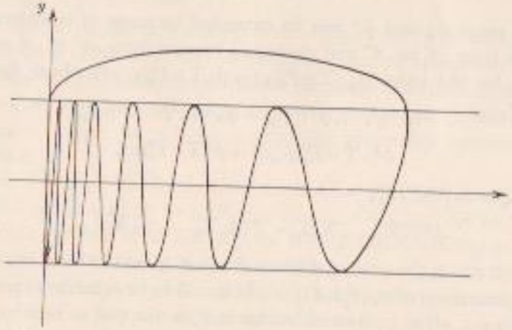


FIGURE 6. A one-one immersion of R^1 in R^2 which is not an imbedding, obtained by joining smoothly part of the curve $y = \sin(1/x)$ to the curve

$$\{(y, 0);\ -\infty < y < 1\}.$$

When the map ϕ is a C^r $(r \geqslant 1)$ diffeomorphism, ϕ_* maps $T_p(\mathscr{M})$ to $T_{\phi(p)}(\mathscr{M}')$ and $(\phi^{-1})^*$ maps $T^*{}_p(\mathscr{M})$ to $T^*{}_{\phi(p)}(\mathscr{M}')$. Thus we can define a map ϕ_* of $T^r_s(p)$ to $T^r_s(\phi(p))$ for any r, s, by

$$T(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^s, \mathbf{X}_1, \ldots, \mathbf{X}_r)|_p$$
$$= \phi_* T((\phi^{-1})^*\boldsymbol{\eta}^1, \ldots, (\phi^{-1})^*\boldsymbol{\eta}^s, \phi_*\mathbf{X}_1, \ldots, \phi_*\mathbf{X}_r)|_{\phi(p)}$$

for any $\mathbf{X}_i \in T_p$, $\boldsymbol{\eta}^i \in T^*{}_p$. This map of tensors of type (r, s) on $\mathscr{M}$ to tensors of type (r, s) on $\mathscr{M}'$ preserves symmetries and relations in the tensor algebra; e.g. the contraction of $\phi_*\mathbf{T}$ is equal to ϕ_* (the contraction of $\mathbf{T}$).

2.4 Exterior differentiation and the Lie derivative

We shall study three differential operators on manifolds, the first two being defined purely by the manifold structure while the third is defined (see §2.5) by placing extra structure on the manifold.

The *exterior differentiation* operator d maps r-form fields linearly t $(r+1)$-form fields. Acting on a zero-form field (i.e. a function) f, gives the one-form field $\mathrm{d}f$ defined by (cf. §2.2)

$$\langle \mathrm{d}f, \mathbf{X}\rangle = Xf \text{ for all vector fields } \mathbf{X} \qquad (2.$$

and acting on the r-form field

$$\mathbf{A} = A_{ab\ldots d}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d$$

it gives the $(r+1)$-form field $\mathrm{d}\mathbf{A}$ defined by

$$\mathrm{d}\mathbf{A} = \mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d. \qquad (2.$$

To show that this $(r+1)$-form field is independent of the coordinat $\{x^a\}$ used in its definition, consider another set of coordinates $\{x^{a'}$ Then

$$\mathbf{A} = A_{a'b'\ldots d'}\,\mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'},$$

where the components $A_{a'b'\ldots d'}$ are given by

$$A_{a'b'\ldots d'} = \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}.$$

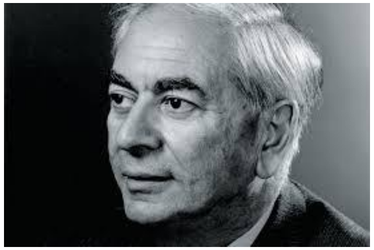
Thus the $(r+1)$-form $\mathrm{d}\mathbf{A}$ defined by these coordinates is

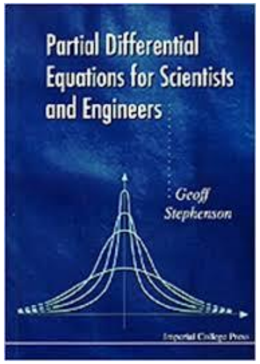
$$\begin{aligned}
\mathrm{d}\mathbf{A} &= \mathrm{d}A_{a'b'\ldots d'}\,\mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'}\\
&= \mathrm{d}\left(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}\right) \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'}\\
&= \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}\mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'}\\
&\quad + \frac{\partial^2 x^a}{\partial x^{a'}\,\partial x^{e'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}\,\mathrm{d}x^{e'} \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'} + \ldots +\\
&= \mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d
\end{aligned}$$

as $\partial^2 x^a/\partial x^{a'}\,\partial x^{e'}$ is symmetric in a' and e', but $\mathrm{d}x^{e'} \wedge \mathrm{d}x^{a'}$ is skew. Not that this definition only works for *forms*; it would not be independer of the coordinates used if the $\wedge$ product were replaced by a tens product. Using the relation $\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$, which holds for arb trary functions f, g, it follows that for any r-form $\mathbf{A}$ and form I $\mathrm{d}(\mathbf{A} \wedge \mathbf{B}) = \mathrm{d}\mathbf{A} \wedge \mathbf{B} + (-)^r \mathbf{A} \wedge \mathrm{d}\mathbf{B}$. Since (2.8) implies that the loc coordinate expression for $\mathrm{d}f$ is $\mathrm{d}f = (\partial f/\partial x^i)\,\mathrm{d}x^i$, it follows th $\mathrm{d}(\mathrm{d}f) = (\partial^2 f/\partial x^i\,\partial x^j)\,\mathrm{d}x^i \wedge \mathrm{d}x^j = 0$, as the first term is symmetric an the second skew-symmetric. Similarly it follows from (2.9) that

$$\mathrm{d}(\mathrm{d}\mathbf{A}) = 0$$

holds for any r-form field $\mathbf{A}$.





Partial Differential Equations for Scientists and Engineers

Geoff Stephenson

Imperial College Press

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On the Magnetic Flux Linkage of an Electrically-Conducting Fluid: A Treatment of the Relativistic Case Using the Exterior Calculus Formalism

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(*Received December 11, 1978*)

Hide's (1979) theorem on the magnetic flux linkage of an electrically-conducting fluid is extended to the fully general relativistic case by rederiving the theorem in the elegant and succinct formalism of Cartan's exterior calculus.

In the previous paper, Hide (1979) derives an electromagnetic theorem which provides a more rigorous footing for his method for determining the radius of a planetary fluid core from observations of the magnetic field near the planetary surface. The theorem states that the time rate of change of the quantity N, defined as the number of intersections of magnetic lines of force with a closed material surface S, is equal to minus twice the integral of the current density divided by the electrical conductivity around all the lines of S where the magnetic field is tangential to S.

In the present paper I note that in the mathematical formalism of Cartan's exterior calculus, both the equations of electromagnetism, and the theorems of Gauss and Stokes can be elegantly and succinctly expressed. This formalism is probably not familiar to many geophysicists, and to demonstrate its conciseness I shall give a derivation of Hide's theorem in the exterior calculus. Since the formalism is manifestly Lorentz invariant, the following may be thought of as the fully relativistic extension of Hide's

p. 79], an antisymmetric matrix, in local coordinates, whose components are related to the electric and magnetic fields. Referring to Eq. (n) of Hide's paper by (Hn), then Gauss's and Faraday's laws (H3, H8) together are

$$d\mathbf{F}=0. \tag{2}$$

Consider a spacelike 2-surface S with spherical topology. Putting $\mathbf{F}$ in (1) with $\partial P=S$, and using (2), gives

$$\int_S \mathbf{F}=0. \tag{3}$$

Now, defining $S(+)$ as that subset of S which maximises

$$\int_m \mathbf{F}, \quad m\subset S, \tag{4}$$

then the definition of N given by

$$N\equiv\int_{S(+)}\mathbf{F} \tag{5}$$

is equivalent to Hide's definition (H1).

Denote the vector field generating the time-mapping of $S(+)$ by $\mathbf{V}$. Hide considers the case when S is a material surface (i.e. $\mathbf{V}=\mathbf{U}$), where $\mathbf{U}$ is the fluid 4-velocity, but in what follows I treat the more general case $\mathbf{V}\neq\mathbf{U}$. The time rate of change of N is given by its Lie derivative along $\mathbf{V}$, i.e.

$$\frac{dN}{dt}=\mathscr{L}_{\mathbf{V}}\int_{S(+)}\mathbf{F}. \tag{6}$$

Since by construction this integral is invariant under the mapping generated by $\mathbf{V}$, it follows that

$$\mathscr{L}_{\mathbf{V}}\int_{S(+)}\mathbf{F}=\int_{S(+)}\mathscr{L}_{\mathbf{V}}\mathbf{F}. \tag{7}$$

An arbitrary p-form $\mathbf{G}$ satisfies the identity

$$\mathscr{L}_{\mathbf{V}}\mathbf{G}=\mathbf{V}\lrcorner d\mathbf{G}+d(\mathbf{V}\lrcorner\mathbf{G}), \tag{8}$$

where $\lrcorner$ denotes a contraction (in local coordinates $\mathbf{V}\lrcorner\mathbf{G}=pV^aG_{a\ldots b}$). Substituting $\mathbf{F}$ for $\mathbf{G}$ in (8), then (2), (6), (7) and the generalised Stokes's theorem (1), give

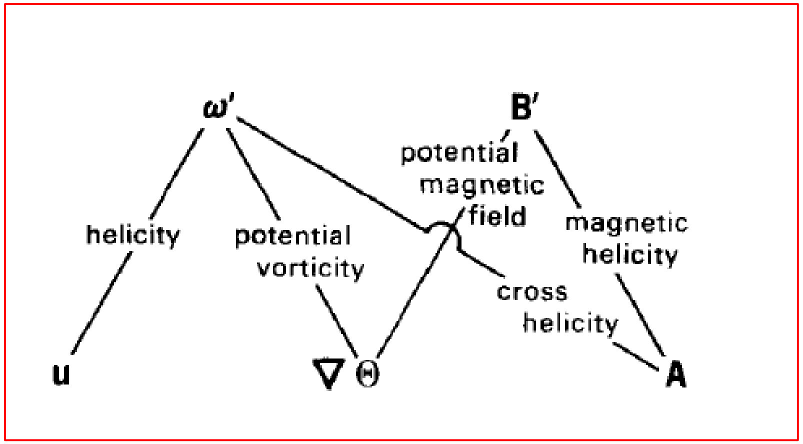
$$\frac{dN}{dt}=\int_{\partial S(+)}\mathbf{V}\lrcorner\mathbf{F}. \tag{9}$$

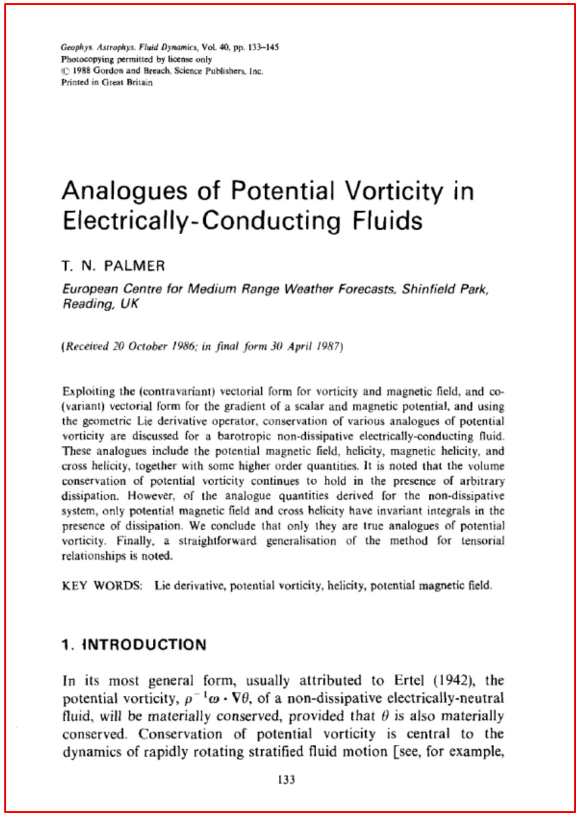
since g is linear and $\mathbf{V}$ is normal to $\partial S(+)$.

When $\mathbf{W}=0$, then (14) becomes

$$\frac{dN}{dt}=-2\int_{\partial S(+)}\sigma^{-1}\mathbf{J}, \tag{15}$$

which is equivalent to (H12) and shows that (H12) is still valid for the

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Analogues of Potential Vorticity in Electrically-Conducting Fluids

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Exploiting the (contravariant) vectorial form for vorticity and magnetic field, and co-(variant) vectorial form for the gradient of a scalar and magnetic potential, and using the geometric Lie derivative operator, conservation of various analogues of potential vorticity are discussed for a barotropic non-dissipative electrically-conducting fluid. These analogues include the potential magnetic field, helicity, magnetic helicity, and cross helicity, together with some higher order quantities. It is noted that the volume conservation of potential vorticity continues to hold in the presence of arbitrary dissipation. However, of the analogue quantities derived for the non-dissipative system, only potential magnetic field and cross helicity have invariant integrals in the presence of dissipation. We conclude that only they are true analogues of potential vorticity. Finally, a straightforward generalisation of the method for tensorial relationships is noted.

KEY WORDS: Lie derivative, potential vorticity, helicity, potential magnetic field.

1. INTRODUCTION

In its most general form, usually attributed to Ertel (1942), the potential vorticity, $\rho^{-1}\omega\cdot\nabla\theta$, of a non-dissipative electrically-neutral fluid, will be materially conserved, provided that θ is also materially conserved. Conservation of potential vorticity is central to the dynamics of rapidly rotating stratified fluid motion [see, for example,

Let G denote a rank *(m,0)* tensor and A an *m-1* form, satisfying

$$\mathcal{L}_U G = 0 \qquad \mathcal{L}_U A = 0$$

then (because exterior derivatives commute with Lie derivatives):

$$\frac{dS}{dt} = 0$$

where

$$S = G.dA$$

is a scalar.

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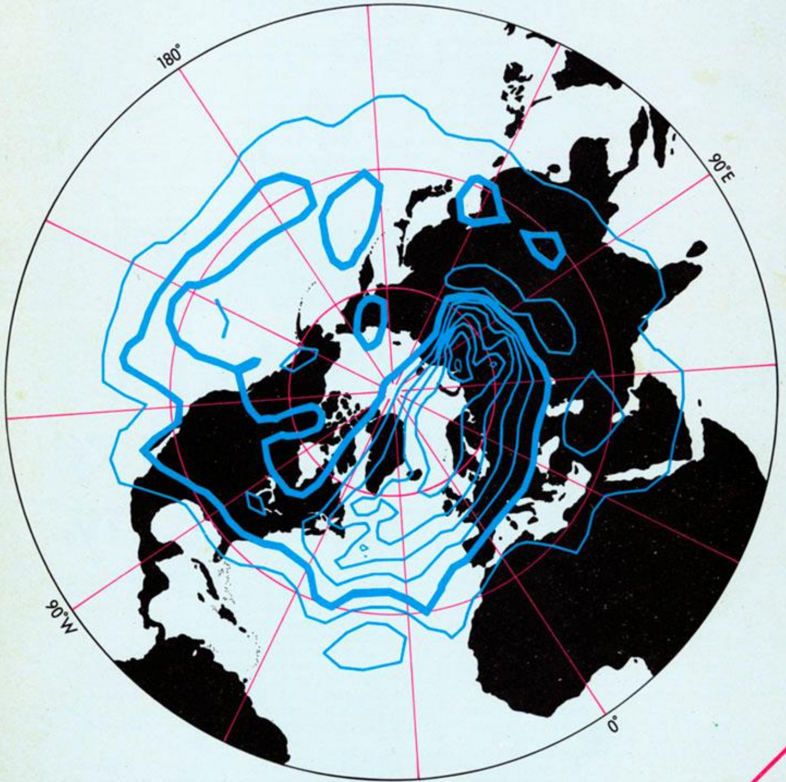

PLANETARY WAVES IN THE STRATOSPHERE

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