Computational models of magnetic field generation in the Earth

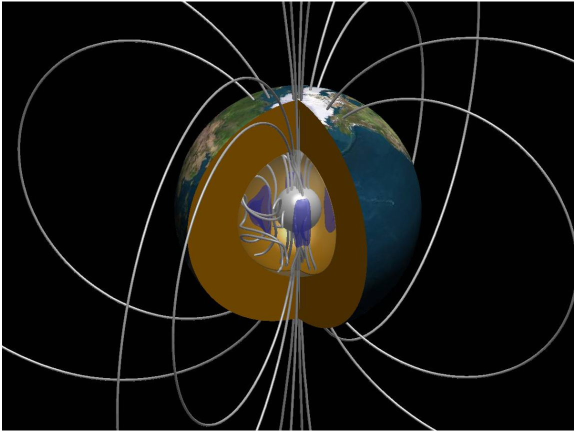

Image: J. Aubert

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Chris Finlay (DTU Copenhagen)
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Summary

Part I

- We report results of numerical dynamo simulations
- These are in the regime where the Lorentz force plays a leading order role
- We develop a new relation between energy and dissipation
- The prediction for the dissipation in the Earth's core (2TW) is sensible — but relies on a huge extrapolation

Part II

- I report on progress towards a complementary theory, in which viscosity is zero
- Introduce a new method for computing mean field dynamos
- Extendable to 3-D convective dynamos

Preamble: Energies and dissipation in Earth

For Earth: at core surface $B \sim 0.2\text{mT} \rightarrow 2\text{mT}$ in interior

$$E_M = \frac{B^2}{2\mu_0} \sim 0.1 - 10 \quad SI$$

For Earth: at core surface $V \sim 0.5\text{mm/sec}$

$$E_V = \frac{\rho v^2}{2} \sim 10^{-3} \quad SI$$

So $E_M \gg E_K$ by $10^2 - 10^4$

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Two important numbers:

Ekman
Magnetic Prandtl

Diffusivity ratio

$$\text{Magnetic Prandtl number} = \frac{\text{Rate of diffusion of momentum}}{\text{Rate of diffusion of B-field}} = \frac{\nu}{\eta}$$

In metals this is 10^{-5} or less.
Can lead to scale separation between velocity and magnetic field:
Length scale of $\mathbf{B}$ $>$ Length scale of $\mathbf{v}$

Ekman number

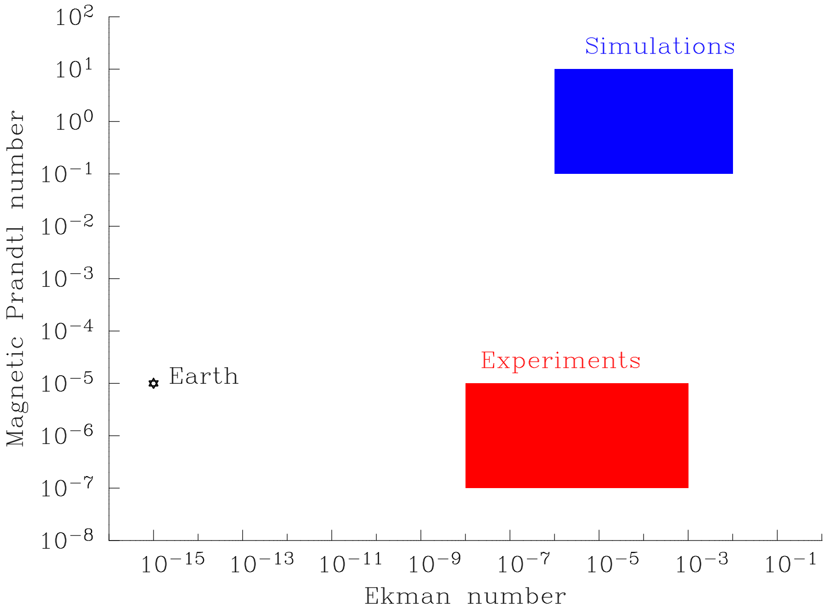
$$\text{Ekman number} = \frac{\text{Viscous force}}{\text{Coriolis (rotational) force}}$$

- $E = \nu / 2L^2\Omega$ [ν=viscosity; L=length scale; Ω=rotation rate]

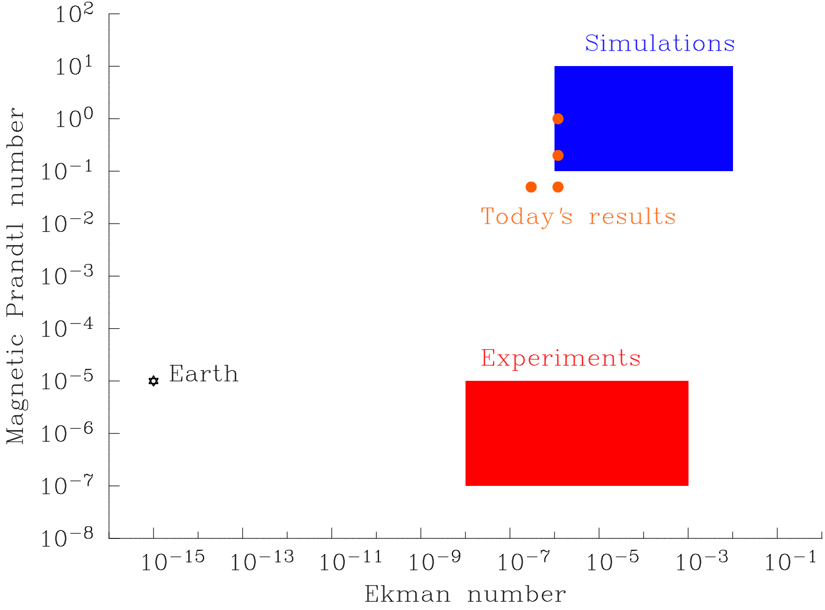
e.g. Ekman number of

- mantle $E \sim 10^{10}$
- bath of water $E \sim 1$
- oceans $E \sim 10^{-9}$
- Earth's core $E \sim 10^{-15}$
- Numerical simulations: $E \sim 10^{-5} - 10^{-7}$

Parameter Space

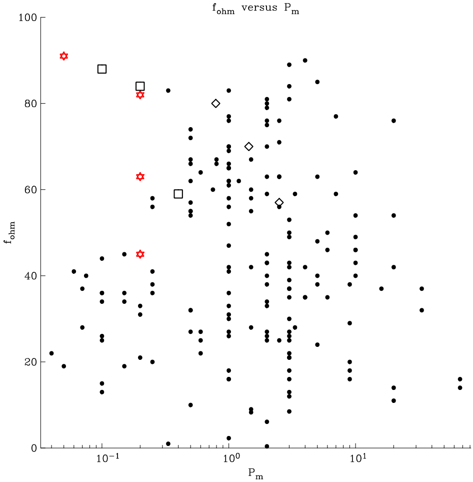


Parameter Space



In our models dissipation is Ohmically dominated

Ohmic fraction



Magnetic Prandtl

The governing equations I.

We begin with the dimensional equations describing core dynamics under the MHD and Boussinesq approximations:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\rho_0\Omega(\hat{\boldsymbol{z}} \times \mathbf{u}) = -\nabla P + \rho_0 \alpha g_0 T \hat{\boldsymbol{r}} + \frac{1}{\mu_0}((\nabla \times \mathbf{B}) \times \mathbf{B}) + \rho_0 \nu \nabla^2 \mathbf{u} \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (2)$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T + h \quad (3)$$

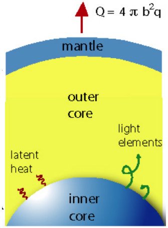
Non-dimensionalised Navier-Stokes and Induction

$$\frac{E}{Pm}\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\right) + \hat{\mathbf{z}}\times\boldsymbol{u} = -\,\boldsymbol{\nabla}\Pi + R_a\,Pm\,T\,\mathbf{r} + E\nabla^2\boldsymbol{u} + [\boldsymbol{\nabla}\times\mathbf{B}]\times\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\mathbf{B}) + \nabla^2\mathbf{B},$$

with $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0$, where $\boldsymbol{u}$ denotes the core flow, Π the modified pressure, $\mathbf{B}$ the magnetic field and T the temperature.
Pr=1
Augment with equation of heat transport

Numerical calculations

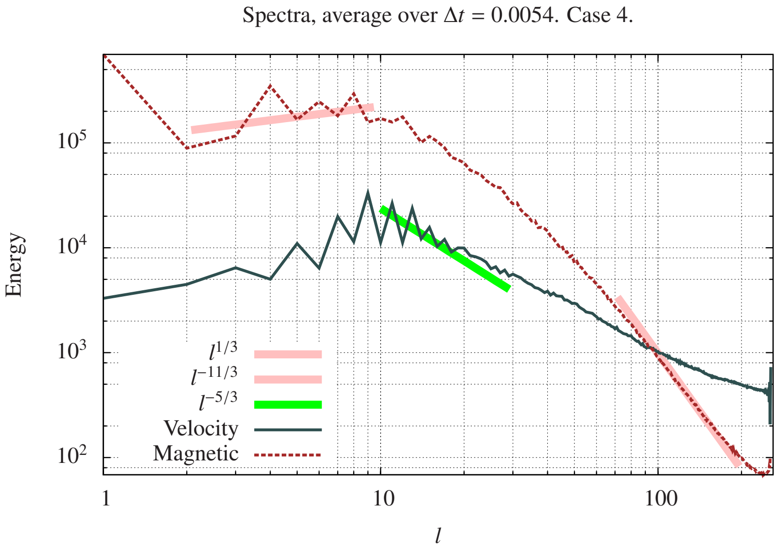


- We use a variety of Rayleigh numbers
- Prandtl number is always unity
- We follow Sakuraba & Roberts and use half internal heating (secular cooling) and half heat flux at the inner core boundary
- The heat flux is constant at the CMB; horizontal temperature variations are able to develop there
- Inner core is electrically conducting
- Mantle is electrically insulating
- Non-slip boundary conditions on velocities

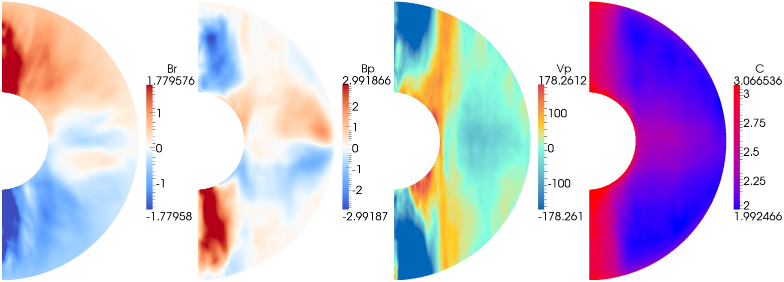
The Simulations

Case	Ekman	Rayleigh	q=Pm	Pr
S0	1.18E-06	219.7	0.20	1.0
S1	1.18E-06	1098.5	0.20	1.0
S2	1.18E-06	6591.0	0.20	1.0
S3	1.18E-06	219.7	1.00	1.0
S4	2.95E-07	6591.0	0.05	1.0

Sheyko's scale separation; E=3×10^{-7}; Pm=0.05



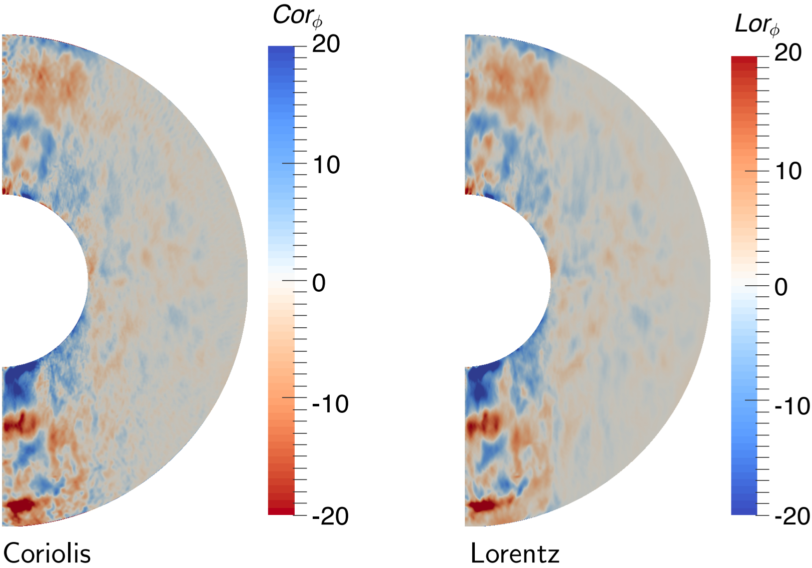
A hot tangent cylinder



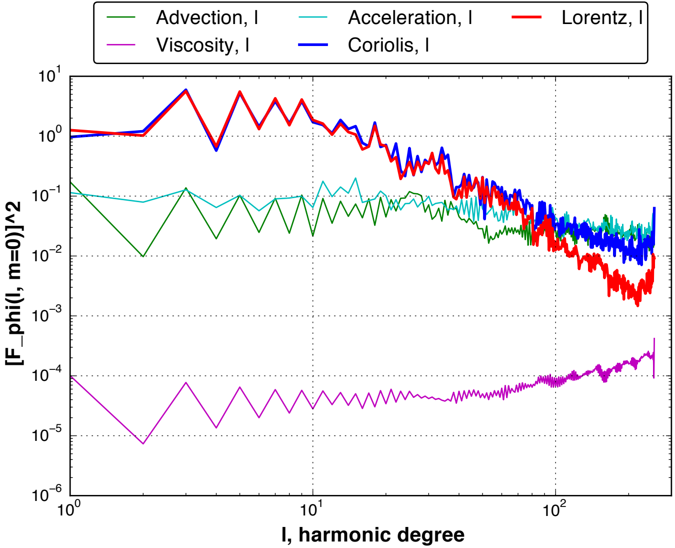
(a) B_r (b) B_φ (c) U_φ (d) T

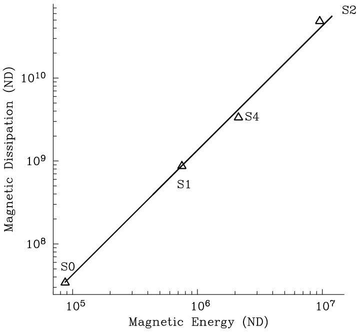
Meridional sections of ϕ- and time- averaged fields. Case 4.

Longitudinally-averaged ϕ-force balance (S4)

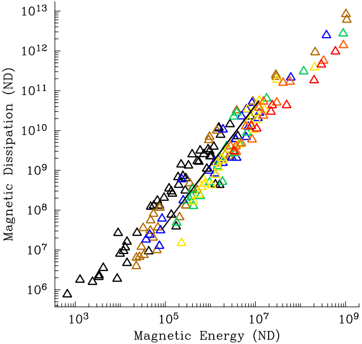


Spectrum shows magnetostrophic force balance at large scales





a)



b)

Solid line $D = 1.35E_M^{3/2}$. Models S1-3 have same Pm and E.

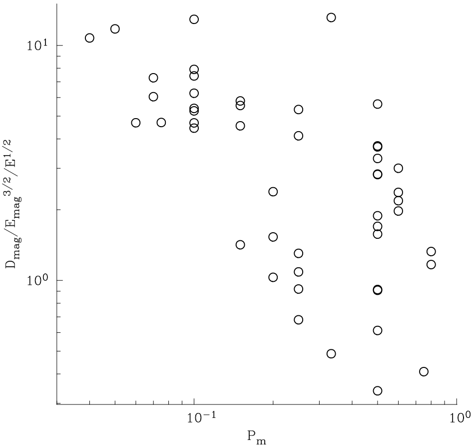
D=Magnetic dissipation

E_M=Magnetic energy

S4 has four times lower Pm and E.

A regression on E, P_m weakly predicts Ohmic dissipation in Earth of 2TW.

Reduced Ohmic dissipation



Magnetic Prandtl

Summary

- We achieved a balance Coriolis $\sim$ Lorentz $\sim$ Pressure $\sim$ Buoyancy
- Strong fields are inside tangent cylinder
- Empirical scaling law: $D \sim E_M^{3/2}$
- Dissipation rises as Pm is reduced... counteracted by a decrease as Ekman number is dropped
- Suggests $\sim$2TW of Ohmic dissipation (c.f. 44TW leaving whole Earth)
- Computer time: 1700 CPU years

A Revived Approach

The magneto-hydrodynamics of a rotating fluid and the earth's dynamo problem

By J. B. Taylor

U.K.A.E.A., Culham Laboratory, Abingdon, Berkshire

(*Communicated by Sir Edward Bullard, F.R.S.—Received* 14 *February* 1963)

This paper discusses a rotating, incompressible fluid enclosed within a rigid boundary which is a surface of revolution. It is shown that if viscous forces are negligible, then, in the presence of magnetic fields, the fluid can execute slow, steady relative motions only if the magnetic force satisfies a constraint. In cylindrical polar co-ordinates this constraint can be written

$$\int_{r=r_0} (\mathbf{j} \times \mathbf{B})_\phi \, \mathrm{d}\phi \, \mathrm{d}z = 0;$$

that is, the couple exerted by the magnetic forces on any cylinder of fluid coaxial with the axis of rotation must vanish.

Furthermore, subject to certain restrictions on the shape of the container (which, for example, are fulfilled by a sphere but not by a cylinder), it is shown that if the field satisfies the above condition then the fluid velocity is completely determined by the instantaneous value of the magnetic field (together with that of the density if buoyancy forces are important). This velocity is such that the necessary conditions on the field will continue to be satisfied. An algorithm for the determination of the velocity is given and its application to the earth's dynamo problem is indicated.

Navier-Stokes and Induction

[Non-dimensionalised]

$$R_o\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}\right) + \hat{\mathbf{z}}\times\mathbf{u} = -\,\boldsymbol{\nabla}\Pi + C\hat{\mathbf{r}} + E\nabla^2\mathbf{u}$$
$$+\,[\boldsymbol{\nabla}\times\mathbf{B}]\times\mathbf{B},$$
$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \nabla^2\mathbf{B},$$

[+ equation of heat transfer]

with $\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0$, where $\mathbf{u}$ denotes the core flow, Π the modified pressure, $\mathbf{B}$ the magnetic field and C the buoyancy force.
$R_o \sim 10^{-9}$ (Rotation/Magnetic Decay)
$E \sim 10^{-15}$

Magnetostrophic Balance

[Non-dimensionalised]
Slow motions

$$\hat{\mathbf{z}} \times \mathbf{u} = -\,\nabla\Pi + \quad C\hat{\mathbf{r}} \quad + [\nabla \times \mathbf{B}] \times \mathbf{B},$$

Coriolis   Pressure   Buoyancy   Lorentz

The onset of convection

- Slow motions governed by Proudman-Taylor theorem:

$$\frac{\partial \mathbf{u}}{\partial z} = 0 (\text{z parallel to } \Omega)$$

- Viscosity breaks P-T and allows convection to occur

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Repercussions of $Ro = E = 0$

- NOT the Euler equation for velocity
- Seductive: no viscosity removes the need to resolve thin boundary layers whilst the absence of inertia filters out very short time scales
- Magnetic forces provide a mechanism by which the Taylor-Proudman theorem can be broken, and convection is then permitted even in the absence of viscosity
- Dissipation by Joule heating (Ohmic dissipation), provides a mechanism to remove small scales, allowing the system to attain a large-scale equilibrium
- Dispersion relation: inertial waves removed, only the so-called MC branch remains — GOOD!
- Similar filtering to $\nabla \cdot \mathbf{u} = 0$: no sound waves

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The Taylor State (Taylor [1963])

This truly makes viscosity unimportant (E=Ro=0) [Dimensional]
Integrate

$$\begin{array}{ccccccccc} 2\rho\boldsymbol{\Omega} \wedge \mathbf{v} & = & -\nabla p & + & \mathbf{J} \wedge \mathbf{B} & + & \rho' \mathbf{g} \\ \text{Coriolis force} & = & -\text{Pressure Gradient} & + & \text{Lorentz Force} & + & \text{Buoyancy} \end{array}$$

over cylinders coaxial with rotation axis; find

$$\int_{C(s)} [\mathbf{J} \wedge \mathbf{B}]_\phi \, d\phi \, dz = 0 \qquad \forall s$$

Applies on *every* cylinder.
Taylor showed that when this condition is satisfied, the flow in the core can be uniquely found. [It is necessary and sufficient].

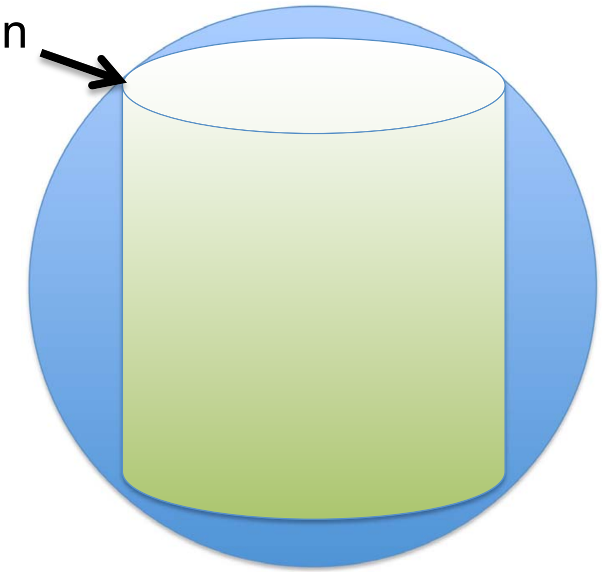
There is a degree of freedom to allow the Taylor State to be achieved.
It is the geostrophic flow $U_G(s)$.

One difficulty

The need for 2 coordinate systems has historically caused problems

One difficulty

RH: Friction



Spherical: mechanical boundary conditions on v; insulating boundary conditions on B

Cylindrical: integration domain for B field

The need for 2 coordinate systems has historically caused problems

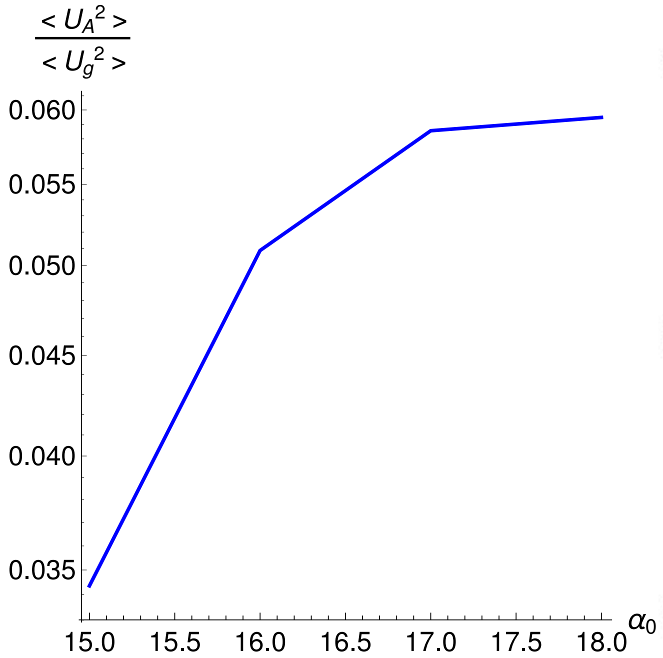
The dynamical case

$$\frac{\partial B}{\partial t} = \nabla \wedge (u \wedge B + \alpha B) + \eta \nabla^2 B$$

$$z \wedge u_m = -\nabla p + J \wedge B$$

$$u = u_m + u_g$$

- $\frac{\int u_A^2 d\ V}{\int u_g^2 d\ V}$ as a function of α_0 in dipole symmetry.

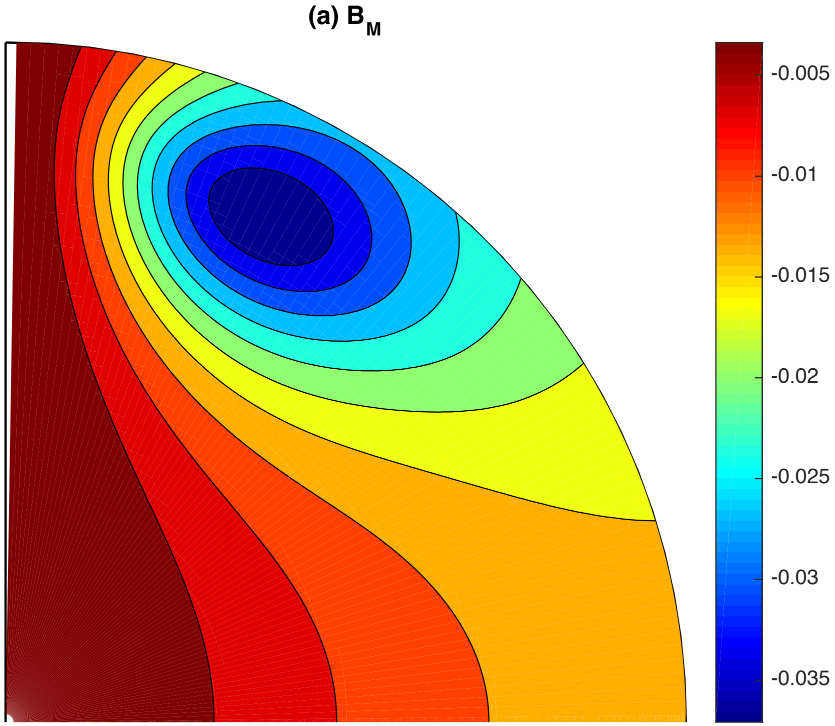


Forcing ->

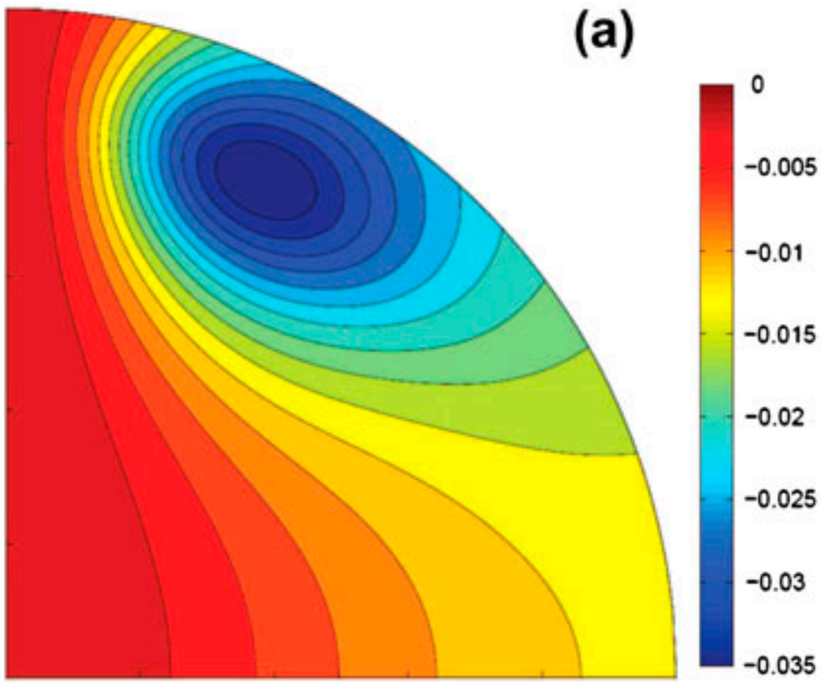
Ratio of geostrophic energy to ageostrophic: Geostrophic component dominant

Meridional B field at saturation

- dipole symmetry with $\alpha_0 = 14.5$



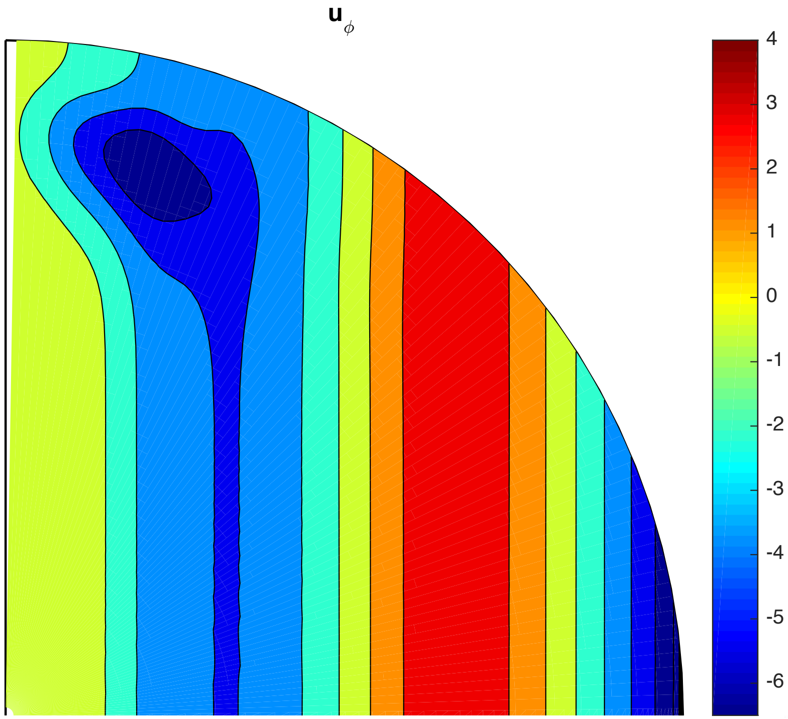
Our solution



Wu & Roberts, (2014)

Total azimuthal flow at saturation; geostrophic component dominates

- dipole symmetry with $\alpha_0 = 14.5$



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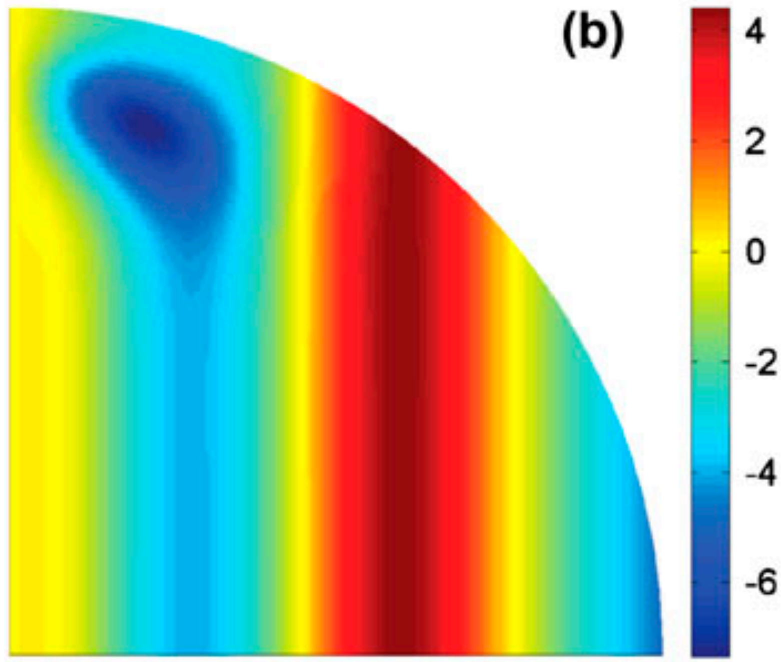

Wu & Roberts, (2014)

The rewards of filtering waves

- Time step controlled only by Courant-Friedrichs-Lewy condition: dt<dx/v
- No waves (Alfven, inertial) to resolve

- Example:
- If $Ro=10^{-9}$, as in Earth, then need $O(10^{10})$ time steps to see if the dynamo is self-exciting by the conventional route

Conclusions

- Taylor's (1963) inviscid ideas can be revived
- Lorentz force active at zeroth order
- Technical solution: stable time integration
- Our approach allows dynamics of timescales of O(100)+ years to be modelled with viscosity having no effect
- We demonstrated the slow and fast manifold relationship
- Have many mean-field examples
- Potentially new types of 3-D thermal dynamo solutions lay ahead: uncharted territory