Space Physics Advanced Option: Waves and Seismology of the Sun

Problem sheet

1. (a) For small-scale waves that do not greatly sense the spherical geometry of the Sun, it is reasonable to approximate the solar interior as a plane-parallel envelope with sound speed $c(z)$ and adiabatic exponent $\gamma(z)$ under constant gravitational acceleration g. When the envelope undergoes adiabatic pulsations with frequency ω, show that the vertical component of the momentum equation may be written

$$\omega^2 w = -\frac{\partial}{\partial z}\left(gw + c^2\chi\right) - \left(c^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} - g\right)\chi ,$$

where w is the vertical component of the velocity $\mathbf{u}$, $\chi = \nabla\cdot\mathbf{u}$ and z is measured downwards. Show also that the vorticity $\nabla\times\mathbf{u}$ is horizontal. Suppose the oscillation variables to have dependence $\exp(ikx + i\omega t)$ on the horizontal coordinate x and time t, and to be independent of the other horizontal coordinate. Show that the only nonzero component η of vorticity satisfies

$$\omega^2\eta = ik\left(c^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} - g\right)\chi .$$

(b) By using the fact that

$$\nabla^2 w = \frac{\partial\chi}{\partial z} - ik\eta$$

(convincing yourself that the equation is correct), or otherwise, derive the equation

$$\frac{\partial^2\chi}{\partial z^2} + \frac{\mathrm{d}\ln(\rho c^4)}{\mathrm{d}z}\frac{\partial\chi}{\partial z} + \frac{1}{c^2}\left\{\left(\omega^2 - k^2c^2\right)\left(1 - \frac{N^2}{\omega^2}\right) + N^2 + \frac{\mathrm{d}}{\mathrm{d}z}\left[c^2\frac{\mathrm{d}\ln(\rho c^2)}{\mathrm{d}z}\right]\right\}\chi = 0 ,$$

where $N(z)$ is the buoyancy frequency. Now set $\chi = v\Psi$, and choose v such that

$$\frac{\partial^2\Psi}{\partial z^2} + K^2\Psi = 0 . \qquad (*)$$

Show that $K^2(z)$ may be written in the form

$$K^2 = \frac{\omega^2 - \omega_\mathrm{c}^2}{c^2} + k^2\left(\frac{N^2}{\omega^2} - 1\right) .$$

What is $\omega_\mathrm{c}(z)$?

2. Starting from equation $(*)$ of Q. 1, explain briefly how the equation

$$\int_{z_1}^{z_2} K\,\mathrm{d}r \simeq \left(n - \frac{1}{2}\right)\pi \qquad (\dagger)$$

is derived and indicate in particular the meaning of z_1 and z_2.
Consider a plane-parallel polytrope of index μ (so $p \propto \rho^{1+1/\mu}$), which might be an appropriate model for the outer layers of the Sun. Making suitable approximations, which should be stated, show that equation $(\dagger)$ yields the dispersion relation

$$\frac{(\mu+1)}{\Gamma_1}\sigma^4 - 2(n+\alpha)\sigma^2 + \mu - \frac{(\mu+1)}{\Gamma_1} = 0 ,$$

where $\sigma^2 = \omega^2/gk$ and α is a constant to be determined. Derive an expression for p-mode frequencies by considering the appropriate root of the dispersion relation. Write down a simplified form of the expression in the high-n limit.

[Hint: you may use without proof the result

$$\int \left(Ax^{-1} - Bx^{-2} - k^2\right)^{1/2} \, \mathrm{d}x \;=\; \left(\frac{A}{2k} - \sqrt{B}\right)\pi \, ,$$

where the integration is between zeros of the integrand, and A, B and k are constants.]

3. Because the Sun's surface temperature and density are both much smaller than their values in the deep interior, it is reasonable in modelling the Sun's radial oscillations to make the approximation that density ρ_0 and p_0/ρ_0 (where p is pressure) both vanish at the surface $r = R$. Starting from the fluid equations, derive the governing equation for linear adiabatic radial oscillations in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\Gamma_1 p_0 r^4 \left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)\right] + \xi\left(\omega^2 \rho_0 r^4 + r^3 \frac{\mathrm{d}(3\Gamma_1 - 4)p_0}{\mathrm{d}r}\right) \;=\; 0 \, ,$$

where $\xi(r)$ is such that the radial velocity is given by

$$u_r \;=\; r\xi \, \exp(i\omega t) \, .$$

Derive a boundary condition that ξ and its derivative must satisfy at $r = R$. Derive also an appropriate boundary condition at the centre of the star, $r = 0$. Explain how this leads to an eigenvalue problem for the frequencies of radial oscillation of the star. Explain without any detailed calculation how the behaviour of the eigenfunctions ξ varies as ω increases.

4. Describe the main characteristics of fast, slow and Alfvénic magneto-acoustic waves.

   Recent observations by the SUMER instrument on board the SOHO spacecraft have detected what have been interpreted as resonant oscillations of a coronal loop. Take as typical loop values: sound speed $c = 1.5 \times 10^5$ m/s, Alfvén speed $v_A = 10^6$ m/s, slow-wave speed $c_{\mathrm{slow}} = 1.5 \times 10^5$ m/s and loop length 50×10^6 m. Is this oscillation a fast, slow or Alfvén mode?

mjt – 17 Oct 2002

Note
Whilst I have endeavoured to eliminate all errors (both conceptual and typographic) from these questions, the bright student should regard identifying and rectifying any remaining errors as part of the set task.