Space Physics Handout 4 : Frozen-in flux, conservation equations

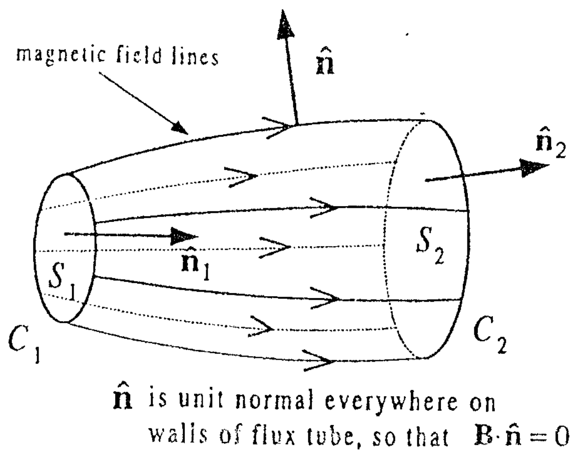
Conservation of magnetic flux in a magnetic flux tube

If C_1 is a closed line in space, and S_1 is a surface subtended by C_1, magnetic field lines which are along C_1 define a magnetic field tube, or magnetic flux tube. (The restriction on defining C_1 is that, if $\underline{dl}_1$ is an infinitesimal vector tangent to C_1, then $\underline{B} \times \underline{dl}_1 \neq 0$ at all points on C_1). The magnetic flux through S_1 is

$$\Phi_1 = \int_{S_1} \underline{B} \cdot \hat{\underline{n}}_1 \, dS_1$$

where $\hat{\underline{n}}_1$ is the unit normal vector to S_1 and dS_1 is an infinitesimal element of it. The surface of the flux tube is defined by the collections of magnetic field lines which are along C_1. Therefore if $\hat{\underline{n}}$ is a unit normal to the surface S of the flux tube, then $\underline{B} \cdot \hat{\underline{n}} = 0$ everywhere on the surface of the flux tube. Let C_2 be another closed line along the flux tube, defined in the same way as C_1, that is, having the same magnetic field lines along it as C_1. The line C_2 subtends the surface S_2; the magnetic flux through S_2 is

$$\Phi_2 = \int_{S_2} \underline{B} \cdot \hat{\underline{n}}_2 \, dS_2$$



Definition of a magnetic flux tube

Let S_{tot} be the closed surface defined by S_1, S_2 and the surface $\underline{S}$ of the flux tube delimited by C_1 and C_2; the volume enclosed by S_{tot} is defined by V. The magnetic flux Φ_{tot} through S_{tot} is zero, as

$$\Phi_{tot} = \int_{S_{tot}} \underline{B} \cdot \hat{\underline{n}} \, dS_{tot} = \int_V \nabla \cdot \underline{B} \, dV = 0$$

given that $\nabla \cdot \underline{B} = 0$ everywhere.

($\hat{\underline{n}}$ denotes here the – outward pointing – unit normal to S_{tot}, therefore we can set for $\hat{\underline{n}}_1 = -\hat{\underline{n}}$, $\hat{\underline{n}}_2 = \hat{\underline{n}}$ over S_1 and S_2 respectively).

The total flux Φ_{tot} is made up of the fluxes (counted positive outward over S_{tot}) of the fluxes through S_1, S_2 and the surface S of the flux tube:

$$\Phi_{tot} = -\Phi_1 + \Phi_2 + \int_S \underline{B} \cdot \hat{\underline{n}} \, dS = 0$$

where the integral, calculated over the walls of the flux tube is $\int_S \underline{B} \cdot \hat{\underline{n}} \, dS = 0$ by the definition of that surface ($\underline{B}$ being tangent to it everywhere).

We now have the result: $\Phi_1 = \Phi_2$

which shows that the magnetic flux along a flux tube is constant (this is called the strength of the flux tube).

The magnetic induction equation

Express the electric field vector from Ohm's law as:

$$\underline{E} = -\underline{v}\times\underline{B} + \underline{j}/\sigma$$

and substitute it into (M3):

$$\frac{\partial \underline{B}}{\partial t} = -\nabla\times\underline{E} = -\nabla\times\left(-\underline{v}\times\underline{B} + \underline{j}/\sigma\right)$$

We also use (M1): $\nabla\times\underline{B} = \mu_0\underline{j}$

to substitute for the current density vector to get :

$$\frac{\partial \underline{B}}{\partial t} = -\nabla\times\left(-\underline{v}\times\underline{B} + \underline{j}/\sigma\right) = \nabla\times(\underline{v}\times\underline{B}) - \nabla\times(\eta\,\nabla\times\underline{B})$$

where $\eta = \frac{1}{\mu_0\sigma}$ is the magnetic diffusivity. Using the vector identity below

$$\nabla\times(\nabla\times\underline{B}) = \nabla(\nabla\cdot\underline{B}) - (\nabla\cdot\nabla)\underline{B}$$

and noting from (M2) $\nabla\cdot\underline{B} = 0$, we get the magnetic induction equation:

$$\frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{v}\times\underline{B}) + \eta\,\nabla^2\underline{B}$$

This equation controls the evolution of the magnetic field as a function of time. The first term on the right hand side arises from the motion of the plasma perpendicular to the magnetic field. The second term on the right controls the diffusion of the magnetic field, but it is only relevant when the plasma conductivity is finite. The ratio of the first term to the second term on the right hand side of the induction equation is the magnetic Reynolds number defined as

$$R_m = \frac{L_0 v_0}{\eta} = \mu_0\,\sigma\,L_0\,v_0$$

where L_0 and v_0 are the characteristic length scale and characteristic velocity, respectively, in the plasma.

For most space plasmas, $R_m \gg 1$ everywhere, except in the vicinity of current sheets (which often separate plasmas of different origins), so that the form of the induction equation used is

$$\frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{v}\times\underline{B}) \qquad (1)$$

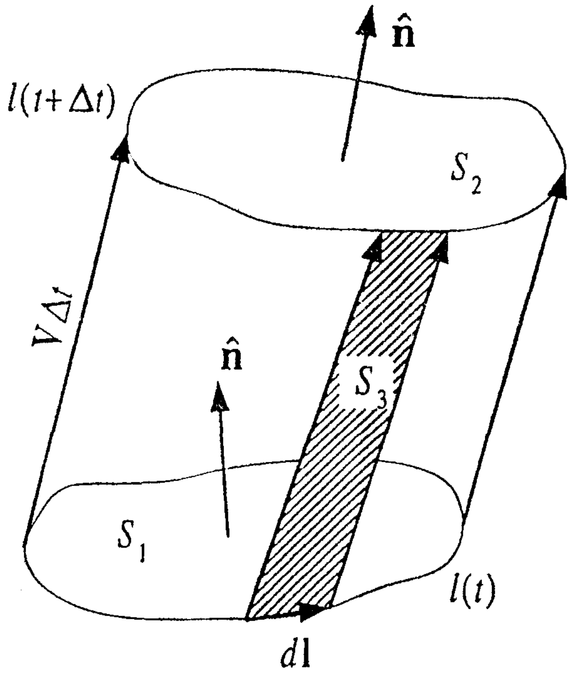
Note that it is this form which is used when Ohm's law is $\underline{E} = -\underline{v}\times\underline{B}$, that is, when the electrical conductivity of the plasma is infinity. This is the magnetohydrodynamic (MHD) approximation; it holds in most space plasmas, except in the vicinity of current sheets.

In the MHD approximation, the magnetic field is "frozen" into the plasma; magnetic flux is carried, unchanged, in a given parcel of plasma moving with velocity $\underline{v}$. Correspondingly, in an infinitely conducting plasma, there is no electric field in the frame moving with the plasma, $\underline{E}$ can only arise as the result of the Lorentz transformation.

In the following, a proof of the "freezing-in" theorem is given, following the proof given by George Sisoce, Solar System Magnetohydrodynamics, 1982.

The magnetic flux, through a closed loop l is given by, as above $\Phi = \int_S \underline{B}\cdot\hat{\underline{n}}\,dS$

where dS is an element of the area on any surface which has l as its perimeter.
Gauss' theorem and Maxwell's equation $\nabla\cdot\underline{B} = 0$ ensure that the magnetic flux is the same through any surface sharing a common perimeter.



Definition of surfaces for the proof of the "freezing in" theorem.

On the figure above, a closed loop of plasma elements is shown at two successive instants, at time t and time t $+\Delta$t. The two closed loops are l(t) and l(t $+\Delta$t). An enclosed volume is formed by the two surfaces S_1 and S_2 that have l (t) and l(t $+\Delta$t) as their perimeters, and the generalised cylinder, S_3, formed by the motion of the closed loop of plasma elements on l, as they move from the position of l(t) by $\underline{v}\Delta t$ to form l(t $+\Delta$t). We call Φ the magnetic flux enclosed by the closed loop of plasma elements on l, and denote by the subscripts 1, 2 and 3 the flux of the magnetic field through the surfaces S_1, S_2 and S_3. If the unit normal vectors to the two surfaces S_1, S_2 are chosen to lie on the same side of each surface with respect to the flow of plasma (as shown in the figure), then the rate of change of the magnetic flux is,

$$\frac{d\Phi}{dt} \; - \xrightarrow{\Delta t \to 0} \; \frac{\Phi_2(t+\Delta t) - \Phi(t)}{\Delta t}$$

because of the condition $\nabla\cdot\underline{B} = 0$, the net magnetic flux through the three surfaces S_1, S_2 and S_3 which form a closed volume must be zero at any time. In particular,

$$-\Phi_1(t+\Delta t) + \Phi_2(t+\Delta t) + \Phi_3 = 0$$

where the negative sign of the Φ_1 term arises because we have to take the outward oriented normal to S_1. As a very important note here, we have used Φ_3 without specifying any time dependence; this is because S_3 is a geometrically defined surface, once S_1, S_2 have been defined. We can eliminate $\Phi_2(t+\Delta t)$ between equations (12) and (13) to give (implicitly for $\Delta t \to 0$):

$$\frac{d\Phi}{dt} = \frac{\Phi_1(t+\Delta t) - \Phi_1(t) - \Phi_3}{\Delta t}$$

We can rewrite this in terms of the flux integrals:

$$\frac{d\Phi}{dt} = \frac{1}{\Delta t}\left\{\int_{S_1}\left[\underline{B}(t+\Delta t) - \underline{B}(t)\right]\cdot\hat{\underline{n}}\,dS - \int_{S_3}\underline{B}\cdot\hat{\underline{n}}\,dS\right\} \quad (2)$$

For the first integral, we have

$$\frac{1}{\Delta t}\int_{S_1}\left[\underline{B}(t+\Delta t)-\underline{B}(t)\right].\hat{\underline{n}}\,dS \xrightarrow{\Delta t\to 0} \int_{S_1}\frac{\partial \underline{B}}{\partial t}.\hat{\underline{n}}\,dS \quad (3)$$

The second integral in (2) can also be converted to an integral over the surface S_1. For this, we write

$$\int_{S_3}\underline{B}.\hat{\underline{n}}\,dS = \int_{l(t)}\underline{B}.\ (d\underline{l}\times\underline{v}\Delta t)$$

because $\hat{\underline{n}}dS = d\underline{l}\times\underline{v}dt$ as can be verified with reference to the figure. We also have, from the rules relating to the mixed cross- and dot products of the three vectors $\underline{B}.\,(d\underline{l}\times\underline{v}\Delta t) = (\underline{v}\times\underline{B}).d\underline{l}\,\Delta t$

so that we get, by applying Stokes' theorem relating line integrals to surface integrals

$$\int_{S_3}\underline{B}.\hat{\underline{n}}\,dS = \int_{l(t)}\underline{B}.(d\underline{l}\times\underline{v}\Delta t) = \int_{l(t)}(\underline{v}\times\underline{B}).d\underline{l}\,\Delta t = \int_{S_1}\left[\nabla\times(\underline{v}\times\underline{B})\right].\hat{\underline{n}}\,dS\,\Delta t \quad (4)$$

Reassembling equation (2), using (3) and (4), we get

$$\frac{d\Phi}{dt} = \int_{S_1}\frac{\partial \underline{B}}{\partial t}.\hat{\underline{n}}\,dS - \frac{1}{\Delta t}\int_{S_1}\left[\nabla\times(\underline{v}\times\underline{B})\right].\hat{\underline{n}}\,dS\,\Delta t$$

and by combining the integrals over S_1 :

$$\frac{d\Phi}{dt} = \int_{S_1}\left[\frac{\partial \underline{B}}{\partial t} - \nabla\times(\underline{v}\times\underline{B})\right].\underline{n}\,dS$$

Thus, the frozen-in condition:

$$\frac{d\Phi}{dt} = 0$$

Follows immediately from equation (1).

This means that the plasma within a magnetic flux tube remains always in that flux tube, as the plasma moves. In addition, plasma elements that are linked by a magnetic field line will always remain on that magnetic field as the plasma and magnetic field line move. So that we can say again that in the MHD approximation (when the magnetic Reynolds number $R_m \gg 1$), the plasma and the magnetic field appear to be "frozen together", in other words, the magnetic field lines are "frozen into" the plasma. This result was first obtained by Hannes Alfven in 1942 who later received the Nobel Prize for Physics. A fundamentally analogous result was known in the last century, when Kelvin and Helmholtz derived the conservation of vorticity in fluids

We need to be careful though, although the concept is very useful and we will apply it during the course, it is only correct if the assumptions leading to the MHD approximation are satisfied, and so it must be used with care.

Conservation equations in space plasmas

During lectures we defined the time rate of change of any quantity that moves with a plasma as being given by the material derivative where we follow the notation normally used in fluid mechanics:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}.\nabla$$

Nature of the j x B force

Using Maxwell's equation (M1), we can write:

$$\underline{j}\times\underline{B} = \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\underline{B}.\nabla)\underline{B}$$

This follows from the following general vector identity

$$(\underline{G}.\nabla)\underline{F} = \frac{1}{2}\Big[\nabla\times(\underline{F}\times\underline{G}) + \nabla(\underline{F}.\underline{G}) - \underline{F}(\nabla.\underline{G}) + \underline{G}(\nabla.\underline{F}) - \underline{F}\times(\nabla\times\underline{G}) - \underline{G}\times(\nabla\times\underline{F})\Big]$$

where $\underline{F}$ and $\underline{G}$ are arbitrary vectors. We have substituted $\underline{F} = \underline{G} = \underline{B}$, and also taken into consideration that $\nabla\cdot\underline{B} = 0$.

Introduce $\hat{\underline{b}}$ as the unit vector along the magnetic field line at a specified point (therefore tangent to the field line at that point) and s as the parameter measuring the distance along the field line, so that

$\underline{B} = B\hat{\underline{b}}$ where B is the magnitude of the magnetic field at that point. Then, because $(\underline{B}.\nabla) = B\,\partial/\partial s$

we get,

$$\frac{1}{\mu_0}(\underline{B}.\nabla)\underline{B} = \frac{B}{\mu_0}\frac{\partial}{\partial s}(B\hat{\underline{b}}) = \hat{\underline{b}}\frac{B}{\mu_0}\frac{\partial B}{\partial s} + \frac{B^2}{\mu_0}\frac{\partial\hat{\underline{b}}}{\partial s} = \hat{\underline{b}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\frac{\partial\hat{\underline{b}}}{\partial s}$$

We define $\hat{\underline{n}}$ as the principal normal to the magnetic field, then $\frac{\partial\hat{\underline{b}}}{\partial s} = \frac{\hat{\underline{n}}}{R_c}$ where R_c is the radius of curvature of the field line at that point. Finally we get for the j x B force expression

$$\underline{j}\times\underline{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \hat{\underline{b}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\frac{\hat{\underline{n}}}{R_c}$$

The various terms on the right hand side of this equation were discussed in lectures. Finally the equation of motion of the plasma, taking into account the plasma pressure and the j x B force can be written as :

$$\rho\frac{d\underline{v}}{dt} = -\nabla p + \underline{j}\times\underline{B} = -\nabla(p+p_B) + \hat{\underline{b}}\frac{\partial}{\partial s}(p_B) + \hat{\underline{n}}\left(\frac{2p_B}{R_c}\right)$$

Where the ratio of the plasma pressure to the magnetic pressure is defined as the "plasma beta",

$$\beta = p/p_B = \frac{2\mu_0 p}{B^2}$$

which represents the relative importance of the forces exerted on the plasma by the pressure gradients in the plasma and the magnetic field.

The effects of the magnetic tension and magnetic pressure forces were discussed in lectures. If we think back to frozen-in field effects, where we have field lines moving and convecting with the plasma, in fact plasma flow cannot distort the magnetic field with impunity since the resulting magnetic pressure and tension force will react back on the plasma to change the motion. In general, the magnetic force tends to oppose the plasma motion which tends to cause compression or rarefaction of the field or which tends to twist it up.

Conservation of energy

This can be quite a complex topic and for simplicity it can in general be assumed that the plasma acts as an ideal gas and that the equation of state is

$$\frac{d}{dt}\left(p/\rho^\gamma\right) = 0$$

where γ is the adiabatic exponent.