Space Physics - Problem Sheet 2 Solutions

Question 1

1a) The equation of motion of the particle is (as $\underline{E} = 0$)

$$m \frac{d\underline{v}}{dt} = q \underline{v} \times \underline{B} \qquad (1.1)$$

and the three equations for the components of $\underline{v}$ are:

$$\frac{dv_x}{dt} = \frac{qB}{m} v_y \qquad (1.2)$$

$$\frac{dv_y}{dt} = -\frac{qB}{m} v_x \qquad (1.3)$$

$$\frac{dv_z}{dt} = 0 \qquad (1.4)$$

Take derivative of (1.2):

$$\frac{d^2 v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} + \frac{q}{m} v_y \frac{dB}{dt}$$

$$= \frac{qB}{m}\frac{dv_y}{dt} + \frac{q}{m} v_y \frac{dB}{dy}\frac{dy}{dt}$$

and since $dy/dt = v_y$, we get

$$\frac{d^2 v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} + \frac{q}{m} v_y^2 \frac{dB}{dy} \qquad (1.5)$$

Take derivative of (1.3):

$$\frac{d^2 v_y}{dt^2} = -\frac{qB}{m}\frac{dv_x}{dt} - \frac{q}{m} v_x \frac{dB}{dt}$$

$$= -\frac{qB}{m}\frac{dv_x}{dt} - \frac{q}{m} v_x \frac{dB}{dy}\frac{dy}{dt}$$

$$= -\frac{qB}{m}\frac{dv_x}{dt} - \frac{q}{m} v_x v_y \frac{dB}{dy} \qquad (1.6)$$

Substitute (1.3) into the first term on the RHS of (1.5)

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_x + v_y^2 \frac{q}{m}\frac{dB}{dy} \quad (1.7)$$

which is the equation (1) you were asked to derive. On substituting (1.2) into first term on RHS of (1.6):

$$\frac{d^2 v_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_y - v_x v_y \frac{q}{m}\frac{dB}{dy} \quad (1.8)$$

which is equation (2) from the question, and equation (3) in the question was derived right at the start of the solution is (1.4)

1b) Substitute the Taylor expansion $B(y) = B_0 + (y-y_0)\frac{dB}{dy}$ into the expression $(qB/m)^2$, to get

$$(q/m)^2\left[B_0 + (y-y_0)\frac{dB}{dy}\right]^2 \approx (q/m)^2\left[B_0^2 + 2B_0(y-y_0)\frac{dB}{dy}\right]$$

where we have neglected the second order term in $\left(dB/dy\right)^2$. Using this result in (1.7) gives

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = -\frac{2q}{m}\left[\frac{q}{m}B_0(y-y_0)v_x - \frac{v_y^2}{2}\right]\frac{dB}{dy} \quad (1.9)$$

On the RHS of (1.9) we can substitute the expressions derived in the course for v_x, v_y and $(y-y_0)$:

$$v_x = v_\perp \sin(\Omega t) \quad (1.10) \qquad , \qquad v_y = v_\perp \cos(\Omega t) \quad (1.11)$$

where $\Omega = \frac{qB_0}{m}$ and $v_\perp^2 = v_x^2 + v_y^2$;

$$y - y_0 = \int v_y\, dt = \frac{v_\perp}{\Omega}\sin(\Omega t) \quad (1.12)$$

Calculate $(y-y_0)v_x = \frac{v_\perp^2}{\Omega}\sin^2(\Omega t) = \frac{v_\perp^2}{2\Omega}\left[1-\cos(2\Omega t)\right] \quad (1.13)$

Similarly $v_y^2 = v_\perp^2\cos^2(\Omega t) = \frac{1}{2}v_\perp^2\left[1+\cos(2\Omega t)\right] \quad (1.14)$

Substituting (1.13), (1.14) into (1.9) gives

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = \frac{-q}{m}\frac{v_\perp^2}{2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right] \qquad (1.15)$$

which is equation (4) from the question which we were asked to derive.

Using the result for the Taylor expansion for $(qB/m)^2$ in (1.8) gives

$$\frac{d^2 v_y}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_y = -\frac{2q}{m}\left[\frac{qB_0}{m}(y-y_0)v_y + \frac{1}{2}v_x v_y\right]\frac{dB}{dy} \qquad (1.16)$$

Using (1.10), (1.11) and (1.12) we get:

$$(y-y_0)v_y = \frac{v_\perp^2}{\Omega}\sin(\Omega t)\cos(\Omega t) = \frac{1}{2}\frac{v_\perp^2}{\Omega}\sin(2\Omega t) \qquad (1.17)$$

Similarly $v_x v_y = v_\perp^2 \sin(\Omega t)\cos(\Omega t) = \frac{1}{2}\frac{v_\perp^2}{\Omega}\sin(2\Omega t)$ (1.18)

Substituting (1.17) and (1.18) into (1.16) we get:

$$\frac{d^2 v_y}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_y = -\frac{3}{2}\frac{q}{m}v_\perp^2\frac{dB}{dy}\sin(2\Omega t) \qquad (1.19)$$

1c) Equations (1.15) and (1.19) are those given as (4) and (5) in the question. The first term of the solution of (1.15) is obtained from

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = 0$$

and it is given in (1.10): $v_x = v_\perp \sin(\Omega t)$. The particular solution of (1.15) is obtained by setting $d^2 v_x/dt^2 = 0$, so that

$$\left(\frac{qB_0}{m}\right)^2 v_x = \frac{-q}{m}\frac{v_\perp^2}{2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right]$$

or $v_x = \frac{-q}{m}\left(\frac{m}{qB_0}\right)^2 \frac{v_\perp^2}{2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right]$ so that the complete solution is:

$$v_x = v_\perp \sin(\Omega t) - \frac{1}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right] \qquad (1.20)$$

Similarly to find the solution of the homogenous part of equation (1.19):

$$\frac{d^2 v_y}{dt^2} + \left(\frac{qB_0}{m}\right)^2 = 0$$, which has the solution

$v_y = v_\perp \cos(\Omega t)$. The particular solution of the complete equation is obtained from:

$$\left(\frac{qB_0}{m}\right)^2 v_y = -\frac{3}{2}\frac{q}{m} v_\perp^2 \frac{dB}{dy} \sin(2\Omega t)$$

and we get: $$v_y = -\frac{3}{2}\left(\frac{q}{m}\right)\left(\frac{m}{qB_0}\right)^2 v_\perp^2 \frac{dB}{dy} \sin(2\Omega t)$$

so that the complete solution of equation (1.19) is:

$$v_y = v_\perp \cos(\Omega t) - \frac{3}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy}\sin(2\Omega t) \qquad (1.21)$$

1d) The displacement of the particle in one gyroperiod can be calculated as:

$$\Delta x_T = \int_0^T v_x dt \quad , \quad \Delta y_T = \int_0^T v_y dt \quad \text{where}$$

$$T = 2\pi/\Omega = \frac{2\pi m}{qB}$$

Write Δx_T explicitly:

$$\Delta x_T = \int_0^T v_\perp \sin(\Omega t) dt - \int_0^T \frac{1}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right] dt$$

the terms $\sin(\Omega t)$ and $\cos(2\Omega t)$ integrate to 0 over a gyroperiod, since $\Omega = 2\pi/T$, so that

$$\Delta x_T = -\int_0^T \frac{1}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy} dt = -\frac{1}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy} T$$

For Δy_T we have

$$\Delta y_T = \int_0^T v_\perp \cos(\Omega t) dt - \int_0^T \frac{3}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy}\sin(2\Omega t) dt$$

Both terms integrate to 0, so that $\Delta y_T = 0$ and the average velocities are therefore:

$$\langle v_x \rangle = \frac{\Delta x_T}{T} = -\frac{1}{q}\frac{m v_\perp^2}{2B_0^2}\frac{dB}{dy} \quad , \quad \langle v_y \rangle = 0$$

1e) In the case of the problem as given here, $\underline{B} = B\hat{\underline{z}}$ and $\nabla B = (dB/dy)\,\hat{y}$, so that

$$\underline{B} \times \nabla \underline{B} = -B\left(\frac{dB}{dy}\right)\hat{\underline{x}}$$

In the first approximation, $\underline{B} = B_0\hat{\underline{z}}$ at the particle's gyrocentre, so that

$$\underline{v}_{grad} = \frac{m v_\perp^2}{2qB_0^3}\,\underline{B} \times \nabla \underline{B} = -\frac{m v_\perp^2}{2qB_0^2}\left(\frac{dB}{dy}\right)\hat{\underline{x}}$$

as found above.

1f) The general formula for drift velocity due to a force $\underline{F}$ is

$$\underline{v}_F = \frac{1}{qB^2}\,\underline{F} \times \underline{B}$$

Comparing this with the expression for the drift velocity due to ∇B gives

$$\underline{F}_{grad} = -\frac{m v_\perp^2}{2B}\nabla B$$

Question 2

2a) A particle gyrating around the magnetic field $\underline{B}$ has a gyroradius

$$r_L = \frac{v_\perp}{\Omega} = \frac{m v_\perp}{qB} \quad , \quad \text{where } B = |\underline{B}|$$

The current generated by the particle is

$$I = \frac{q\Omega}{2\pi} = \frac{q^2 B}{2\pi m}$$

That is, the number of times the particle passes through a point on its gyratory trajectory (the gyrofrequency) multiplied by the particle's electric charge. The area of the current is $A = \pi r_L^2$, so that the particle's magnetic moment is

$$\mu = IA = \frac{q^2 B}{2\pi m} \pi \frac{m^2 v_\perp^2}{q^2 B^2} = \frac{m v_\perp^2}{2B}$$

2b) As $\underline{B}$ is cylindrically symmetric, with $B_z \neq 0$, then $B_\phi = 0$, and as $\nabla \cdot \underline{B} = 0$, then

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0, \quad \text{or} \quad \frac{\partial}{\partial r}(r B_r) = -r \frac{\partial B_z}{\partial z}$$

Integrating with respect to r gives

$$r B_r = -\frac{r^2}{2}\frac{\partial B_z}{\partial z}, \quad \text{so that} \quad B_r \simeq -\frac{r}{2}\frac{\partial B}{\partial z}$$

as we assumed $\partial B_z / \partial z \simeq \partial B / \partial z$

2c) The equation of motion of the particle is

$$m \frac{d\underline{v}}{dt} = q \underline{v} \times \underline{B}$$

~~as we assumed that~~ there is no electric field present.

Using $\underline{B} = B_r \hat{\underline{r}} + B_z \hat{\underline{z}}$ so that

$$m \frac{d\underline{v}}{dt} = q B_r \underline{v} \times \hat{\underline{r}} + q B_z \underline{v} \times \hat{\underline{z}}$$

The second term on the RHS of this equation leads to the gyromotion of the particle around the z-axis, as we assumed $B_z \gg B_r$. The additional force on the particle arises from the first term on the RHS,

$$\underline{F}_z = F_z \hat{\underline{z}} = q B_r \underline{v} \times \hat{\underline{r}} = q B_r v_\perp \hat{\underline{z}}$$

where $v_\perp$ is the velocity of the particle perpendicular to $\underline{B}$, as we assumed that $\underline{B}$ is, to a good approximation, along the z axis. This force $\underline{F}_z$ acts on the particle at a distance $r = r_L$ from the z-axis (because that's where the particle is!) with r_L being the gyroradius of the particle. We have, at $r = r_L$

$$F_z = q v_\perp B_r|_{r=r_L} = -q v_\perp \frac{r_L}{2}\frac{\partial B_z}{\partial z}$$

with $r_L = \frac{v_\perp}{\Omega} = \frac{m v_\perp}{qB}$, so that

$$F_z = -\frac{m v_\perp^2}{2B}\frac{\partial B_z}{\partial z} \cong -\frac{m v_\perp^2}{2B}\frac{\partial B}{\partial z} = -\mu \frac{\partial B}{\partial z}$$

2d) The rate of change of the magnetic field as seen by the moving particle (using the approximation $B_z \cong B$) is

$$\frac{dB}{dt} = \frac{\partial B}{\partial z}\frac{dz}{dt} = v_\parallel \frac{\partial B}{\partial z}$$

where $v_\parallel$ is the velocity of the particle along $\underline{B} \cong B_z \hat{z}$.
We next calculate the work done on the particle by the force $\underline{F_z}$.
This is equal to the gain in its kinetic energy in the direction of the force, along z. The work is calculated over one gyroperiod, during which we assume that

$$\frac{\partial B_z}{\partial z} \cong \frac{\partial B}{\partial z} \quad \text{remains (approximately) constant.}$$

The gyroperiod is

$$T = \frac{2\pi}{\Omega} = 2\pi\left(\frac{m}{qB}\right)$$

and the particle travels a distance $v_\parallel T$ in the direction of z during T.

$$\delta W = \int_0^T F_z\, dt = \int_0^{v_\parallel T} F_z\, dz = -\mu \frac{\partial B}{\partial z} v_\parallel T = -\mu \frac{\partial B}{\partial z}\frac{dz}{dt} T = -\mu \frac{dB}{dt} T$$

The average rate of energy gain during the motion of the particle along z is

$$\frac{d}{dt}\left(\tfrac{1}{2} m v_\parallel^2\right) = \frac{\delta W}{T} = -\mu \frac{dB}{dt}$$

2e) As the total kinetic energy remains constant (no electric field is present), its time derivative is zero:

$$\frac{d}{dt}\left(\tfrac{1}{2} m v^2\right) = \frac{d}{dt}\left(\tfrac{1}{2} m v_\perp^2\right) + \frac{d}{dt}\left(\tfrac{1}{2} m v_\parallel^2\right) = 0$$

We have from part 2a) of the question, $\mu = \frac{m v_\perp^2}{2B}$, or $\tfrac{1}{2} m v_\perp^2 = \mu B$, so that

$$\frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) = B\frac{d\mu}{dt} + \mu\frac{dB}{dt}$$

Together with the result from 2d), we have:

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = B\frac{d\mu}{dt} + \mu\frac{dB}{dt} - \mu\frac{dB}{dt} = 0$$

which gives $B\frac{d\mu}{dt} = 0$, but as $B \neq 0 \rightarrow \frac{d\mu}{dt} = 0$, so that the magnetic moment is conserved.

2f) In a time varying magnetic field $\underline{B} = B(t)\,\hat{\underline{z}}$ an electric field is induced (Faraday's law) which is $\nabla \times \underline{E} = -\partial \underline{B}/\partial t$. In the plane perpendicular to $\underline{B}$, and therefore to the z axis, the force acting on the particle is only this induced electric field, therefore the equation of motion is

$$m\frac{d\underline{v}_\perp}{dt} = q\,\underline{E}_\perp \quad ; \text{ taking the dot product with } \underline{v}_\perp \text{ gives:}$$

$$m\underline{v}_\perp \cdot \frac{d\underline{v}_\perp}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) = q\,\underline{E}_\perp \cdot \underline{v}_\perp$$

2g) Using the result from 2f) above, we have (since $\underline{E} = \underline{E}_\perp$)

$$\Delta\left(\frac{1}{2} m v_\perp^2\right) = \int_0^T q\underline{E}\cdot\underline{v}_\perp\, dt = -\oint q\underline{E}\cdot d\underline{l}$$

as a) $\underline{v}_\perp dt = d\underline{l}$, the infinitesimal path length of the particle perpendicular to $\underline{B}$ and

b) the negative sign in front of the last integral arises from the integration (in the right-handed, positive sense) along a complete particle orbit; but the particles orbits in the negative sense (anti-clockwise) for $q>0$, and in the positive sense (clockwise) for $q<0$.

2h) The change in energy, during one gyroperiod, is

$$\Delta\left(\frac{1}{2} m v_\perp^2\right) = -\oint q\underline{E}\cdot d\underline{l} = -q\int_A (\nabla\times\underline{E})\cdot d\underline{A} = q\int_A \frac{\partial \underline{B}}{\partial t}\cdot d\underline{A}$$

where we used Stokes theorem and Faraday's law. Here $d\underline{A}$ is the infinitesimal surface element vector perpendicular to $\underline{B}$ or equivalently, to the z axis.

If we assume that the magnetic field changes by an amount ΔB during one gyroperiod of the particle, then we can substitute $\frac{\partial B}{\partial t} \cong \frac{\Delta B}{T}$ in the integral (noting also that $d\underline{A}$ is parallel to $\underline{B}$),

so that

$$\Delta\left(\tfrac{1}{2} m v_\perp^2\right) = q \frac{\Delta B}{T} A$$

where $A = \pi r_L^2 = \pi \left(\frac{v_\perp m}{qB}\right)^2$ and $T = 2\pi \frac{m}{qB}$,

substituting for T and A gives;

$$\Delta\left(\tfrac{1}{2} m v_\perp^2\right) = \mu \Delta B$$

2i) By the definition of $\mu = \frac{m v_\perp^2}{2B}$, so that

$$\Delta\left(\tfrac{1}{2} m v_\perp^2\right) = \mu \Delta B + B \Delta \mu = \mu \Delta B \text{ from 2h)}$$

and therefore $B \Delta \mu = 0$, and as $B \neq 0$, we have finally

$$\Delta \mu = 0$$

showing that the magnetic moment is conserved also under these assumptions.