Problem Sheet 3 Solutions - Space Physics

Question 1:

1a) The force balance equation $-\nabla p + \underline{j} \times \underline{B} = 0$ shows that in a plasma in steady state in which there are no time variations, the force exerted by the plasma pressure, $-\nabla p$, and the force exerted on the plasma by the Lorentz force (which gives rise to the $\underline{j} \times \underline{B}$ term) are in equilibrium. We also assume there are no other forces acting on the plasma. The $\underline{j} \times \underline{B}$ term can be considered as arising from the combination of the gradient of the magnetic pressure $p_B = B^2/2\mu_0$ perpendicular to the magnetic field lines, and the tension force exerted by the magnetic field.

1b) The current density vector has a z component only (parallel to the axis of the cylinder, and it is

$$\underline{j} = \frac{I}{\pi R^2} \hat{\underline{z}} \quad \text{inside the cylinder} \quad \text{and } \underline{j} = 0 \text{ outside it.}$$

Applying the Biot-Savart law to calculate the magnetic field we get, in cylindrical co-ordinates, $B_r = B_z = 0$ and $B_\phi \neq 0$; or from considerations of symmetry, $B_z = 0$ and $\nabla . \underline{B} = 0$ we also get the same result.

1c) The only non-zero component of the magnetic field, B_ϕ is a function of r only $\partial/\partial\phi = \partial/\partial z = 0$. Applying Ampere's law in its differential form, $\nabla \times \underline{B} = \mu_0 \underline{j}$ we get:

$$\frac{1}{r}\frac{d}{dr}(r B_\phi) = \mu_0 j_z \qquad \text{for} \quad r \leq R$$

$$\frac{1}{r}\frac{d}{dr}(r B_\phi) = 0 \qquad \text{for} \quad r > R$$

Integrating with respect to r gives:

$$rB_\phi = \mu_0 \frac{I}{\pi R^2}\int_0^r r\,dr = \frac{\mu_0 I}{2\pi}\frac{r^2}{R^2} \quad \text{for } r \le R$$

$$rB_\phi = \text{constant} = RB_\phi(R) \quad \text{for } r > R$$

But $B_\phi(R) = \dfrac{\mu_0 I}{2\pi R}$, so that

$$B_\phi(r) = \frac{\mu_0 I}{2\pi R^2}\, r \quad \text{for } r \le R$$

$$B_\phi(r) = \frac{\mu_0 I}{2\pi r} \quad \text{for } r > R$$

Note: using Ampere's law in its integral form gives the same result.

1d) In the force balance equation, $\nabla p = \underline{j} \times \underline{B}$. As $\underline{j} = j_z \hat{\underline{z}}$ and $\underline{B} = B_\phi \hat{\underline{\phi}}$, so that the direction of the $\underline{j}\times\underline{B}$ force is $\hat{\underline{z}} \times \hat{\underline{\phi}} = -\hat{\underline{r}}$, it is directed radially inwards towards the origin, so that

$$\nabla p = \hat{\underline{r}}\frac{dp}{dr} = -j_z B_\phi \hat{\underline{r}} = \frac{-I}{\pi R^2}\frac{\mu_0 I}{2\pi R^2}\, r\hat{\underline{r}} = -\frac{\mu_0}{2}\left(\frac{I}{\pi R^2}\right)^2 r\hat{\underline{r}}$$

We can obtain $p(r)$ by integration:

$$p(r) = \int_0^r \frac{dp}{dr}\,dr = -\frac{\mu_0}{4}\left(\frac{I}{\pi R^2}\right)^2 r^2 + \text{constant}$$

where the integration constant can be determined from the condition $p(R) = p_0$, so that

$$-\frac{\mu_0}{4}\left(\frac{I}{\pi R^2}\right)^2 R^2 + \text{constant} = p_0 \quad \text{or}$$

$$\text{constant} = p_0 + \frac{\mu_0}{4}\left(\frac{I}{\pi R^2}\right)^2 R^2, \text{ so that finally}$$

$$p(r) = p_0 + \frac{\mu_0}{4}\left(\frac{I}{\pi R^2}\right)^2 (R^2 - r^2)$$

1e) The magnetic pressure is $p_B = \frac{B^2}{2\mu_0}$, so that we have, for $r > R$

$$p_B = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)^2 = \frac{\mu_0}{8}\left(\frac{I}{\pi}\right)^2 \frac{1}{r^2}$$

The force balance equation yields $\underline{j} \times \underline{B} = 0$ (the so-called force free configuration) as $p = p_0$ for $r > R$ and $\nabla p = 0$. This means that the force due to the magnetic pressure

$$-\nabla p_B = -\hat{\underline{r}} \frac{d}{dr}(p_B) = \frac{\mu_0}{4}\left(\frac{I}{\pi}\right)^2 \frac{1}{r^3}\hat{\underline{r}}$$

must be balanced by the magnetic tension force, which is therefore

$$\nabla p_B = \hat{\underline{r}} \frac{d}{dr}(p_B) = -\frac{\mu_0}{4}\left(\frac{I}{\pi}\right)^2 \frac{1}{r^3}\hat{\underline{r}}$$

A direct calculation of the magnetic tension force gives the same result (using the formula derived in the course):

$$\frac{(\underline{B}.\nabla)\underline{B}}{\mu_0} = -\frac{1}{\mu_0}\frac{B_\phi^2}{r}\hat{\underline{r}} = -\frac{\mu_0}{4}\left(\frac{I}{\pi}\right)^2 \frac{1}{r^3}\hat{\underline{r}}$$

Question 2:

As $B_r = B_z = 0$, and $B_\phi(r)$ is independent of z, then only the z-component of the curl operator is non-trivial, so we have

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_\phi) = 0$$

$$\Leftrightarrow \frac{\partial}{\partial r}(r B_\phi) = B_\phi + r\frac{\partial B_\phi}{\partial r} = 0,$$ and

dividing through by r gives:

$$\frac{\partial B_\phi}{\partial r} + \frac{B_\phi}{r} = 0.$$

Question 3:

If the scalar function is given as $f(x,y,z)$ then we have in cartesian co-ordinates that

$$\nabla f = \hat{\underline{x}} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{\underline{z}} \frac{\partial f}{\partial z}$$

, where $\hat{\underline{x}}, \hat{y}, \hat{\underline{z}}$ are the unit vectors along the axes of the system, using the definition of the del operator in cartesian co-ordinates $\nabla = \hat{\underline{x}} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{\underline{z}} \frac{\partial}{\partial z}$.

Using the same differential vector operator and the definition of the vector cross product $\underline{A} \times \underline{B} = \hat{\underline{x}}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{\underline{z}}(A_x B_y - A_y B_x)$ where $\underline{A} = A_x \hat{\underline{x}} + A_y \hat{y} + A_z \hat{\underline{z}}$ and $\underline{B} = B_x \hat{\underline{x}} + B_y \hat{y} + B_z \hat{\underline{z}}$ are arbitrary vectors, we are able to derive:

$$\nabla \times (\nabla f) = \hat{\underline{x}} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) + \hat{y} \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) + \hat{\underline{z}} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= 0$$

, because all the 3 components are equal to zero.

Question 4:

The vector $f\underline{A}$ has components of fA_r, fA_θ, fA_ϕ, so that

$$\nabla \cdot (f\underline{A}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (f A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (f A_\phi).$$

Let us evaluate this expression term by term:

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f A_r) = \frac{1}{r^2} \left(2rfA_r + r^2 A_r \frac{\partial f}{\partial r} + r^2 f \frac{\partial A_r}{\partial r} \right)$$

$$= \frac{2fA_r}{r} + A_r \frac{\partial f}{\partial r} + f \frac{\partial A_r}{\partial r} \qquad (1)$$

$$\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (f A_\theta \sin\theta) = \frac{1}{r \sin\theta} \left(A_\theta \sin\theta \frac{\partial f}{\partial \theta} + f \sin\theta \frac{\partial A_\theta}{\partial \theta} + f A_\theta \cos\theta \right)$$

$$= \frac{A_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{f}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{f A_\theta}{r \tan\theta} \qquad (2)$$

$$\frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (f A_\phi) = \frac{1}{r \sin\theta} \left(A_\phi \frac{\partial f}{\partial \phi} + f \frac{\partial A_\phi}{\partial \phi} \right) \qquad (3)$$

Now on the right hand side of the identity we have 2 terms: $f\nabla.\underline{A}$ and $\underline{A}.\nabla f$. The first is explicitly:

$$f\nabla.\underline{A} = f\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]$$

Evaluating this term by term gives:

$$\frac{f}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = \frac{f}{r^2}\left(2rA_r + r^2\frac{\partial A_r}{\partial r}\right) = \frac{2f}{r}A_r + f\frac{\partial A_r}{\partial r} \quad (4)$$

$$\frac{f}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) = \frac{f}{r\sin\theta}\left(\sin\theta\frac{\partial A_\theta}{\partial\theta} + A_\theta\cos\theta\right) = \frac{f}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{fA_\theta}{r\tan\theta} \quad (5)$$

and the third term is $\frac{f}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$ (6)

The second expression on the right hand side can be written as:

$$\underline{A}.\nabla f = A_r(\nabla f)_r + A_\theta(\nabla f)_\theta + A_\phi(\nabla f)_\phi$$

$$= A_r\frac{\partial f}{\partial r} + \frac{A_\theta}{r}\frac{\partial f}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial f}{\partial\phi} \quad (7)$$

Finally, gathering and summing the terms we have evaluated on the left hand side and right hand side of

$\nabla.(f\underline{A}) = f\nabla.\underline{A} + \underline{A}.\nabla f$ proves the identity, since (1) + (2) + (3) = (4) + (5) + (6) + (7)

Question 5:

We are given that at 1 AU, $n_p = 6.6\ cm^{-3} = 6.6\times10^6\ m^{-3}$ and $U_{sw} = 450\ kms^{-1} = 4.5\times10^5\ ms^{-1}$, then the number flux of the protons is $n_p U_{sw} = 6.6\times4.5\times10^{11}\ m^{-2}s^{-1}$

$= 2.97\times10^{12}\ m^{-2}s^{-1}$

The mass flux is the number flux multiplied by the mass of the proton, so it is:

$$n_p U_{sw} m_p = 2.97 \times 10^{12} \times 1.67 \times 10^{-27} \ kg\, m^{-2} s^{-1}$$
$$= 4.96 \times 10^{-15} \ kg\, m^{-2} s^{-1}$$

To obtain the total flux integrated over the surface of the sphere centered at the Sun and passing through the Earth, of radius $R = 1AU = 150 \times 10^6$ km $= 1.5 \times 10^{11}$ m, we have

$$4\pi R^2 n_p U_{sw} m_p = 4\pi \times 2.25 \times 10^{22} \times 4.96 \times 10^{-15} \ kg\, s^{-1}$$
$$= 140.24 \times 10^7 \ kg\, s^{-1}$$
$$= 1.4 \times 10^9 \ kg\, s^{-1}$$
$$= 1,400,000,000 \ kg\, s^{-1} \ !!$$

The quick way to obtain the total mass of electrons lost by the Sun per second, is to divide the figure for protons by the proton-to-electron mass ratio, 1836, this gives $0.76 \times 10^6 \ kg\, s^{-1}$.

Question 6:

6a) The circumferential velocity in the solar equatorial plane at a radius $r_0 = 5R_s$, is

$$5R_s \omega_s = 5 \times 7 \times 10^5 \times 2.86 \times 10^{-6} \ km\, s^{-1} = 10.01 \ km\, s^{-1}$$

If the solar wind is emitted from the Sun at $r_0 = 5R_s$, and therefore receives no further angular momentum, its angular momentum at the source surface distance is conserved: The product of the transverse velocity at the source surface and the radial distance of the source surface (or the angular momentum) is equal to the radial distance of the Earth multiplied by the transverse velocity at the Earth's distance:

$$10.01 \ km\, s^{-1} \times 5 \times 7 \times 10^5 \ km = 1.5 \times 10^8 \ km \times U_{transverse}$$

Hence $U_{transverse} = 0.234 \ km\, s^{-1} \ll 400 \ km\, s^{-1}$, so that it is negligible compared to the solar wind velocity.

6b) Calculate first the orbital rotation rate of the Earth and its orbital velocity:

$$W_E = \frac{2\pi}{365 \times 24 \times 3600} = 2.0 \times 10^{-7} \text{ rad s}^{-1}$$

and $$W_E r_E = 2.0 \times 10^{-7} \text{ rad s}^{-1} \times 1.5 \times 10^{8} \text{ km} = 30 \text{ km s}^{-1}$$

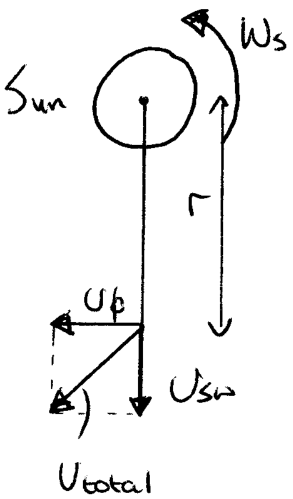
For an Earth-based observer, this represents an apparent transverse velocity (aberration) of the solar wind. The apparent angle with respect to the radial direction (the Earth-Sun line) is

$$\tan^{-1}\left(\frac{30}{400}\right) = 4.3°$$

6c) As the Sun rotates with an angular velocity W_s, the solar wind which is emitted radially (neglecting the very small correction discussed in (6a) above) has a transverse velocity, at a distance r from the Sun, in the frame rotating with the Sun of

$$U_\phi(r) = -W_s r \sin\theta$$, where θ is the polar angle described in lectures.

The total velocity vector can be obtained from the vector addition of the radial component U_{SW} and the azimuthal (or transverse) component $U_\phi(r)$ as shown below: (Note in the solar equatorial plane, $\theta = 90°$, $\sin\theta = 1$).



Streamlines are found from the differential equation:

$$\frac{U_\phi}{U_{SW}} = \frac{-W_s r \sin\theta}{U_{SW}} = \frac{r \sin\theta \, d\phi}{dr}$$

$$\Leftrightarrow \quad dr = \frac{-U_{SW}}{W_s} \, d\phi$$

Integrating we obtain $r - r_0 = \frac{-U_{SW}}{W_s}(\phi - \phi_0)$

This is the equation of an Archimedian spiral (in the equatorial plane of the Sun), representing now the streamlines of the solar wind in the frame of reference rotating with the Sun. In this equation r_0 and ϕ_0 represent the origin of the streamline at the nominal source surface of the solar wind.

6d) Given that the solar wind plasma is infinitely conducting, the magnetic field is frozen into the flow. As the magnetic field is aligned with the solar wind flow velocity at the source surface, it remains aligned with the flow in the reference frame rotating with the Sun.

Therefore the magnetic field lines are aligned with the solar wind streamlines in that frame. The electric field in that frame can be found from the MHD equation

$$\underline{E} = -\underline{v}_{total} \times \underline{B} = 0$$

so $\underline{v}_{total}$ and $\underline{B}$ are aligned in that frame.

6e) The transformation velocity from the rotating to the stationary frame is equal and opposite to the transverse velocity $v_\phi(r)$ of the solar wind in the rotating frame, so it is $\omega_s r$. In this transformation, which "straightens" out the solar wind streamlines (they simply become radial), the magnetic field geometry, in its spiral form, is conserved, as we have assumed that the Lorentz transformation is applicable (none of the velocities are close to the speed of light!) This leaves the magnetic field unchanged between frames of reference. The electric field in this frame of reference is:

$$\underline{E} = -\underline{v}_{sw} \times \underline{B}$$

6f) The equation of the magnetic field lines in the stationary frame is simply that found for the streamlines in the rotating frame. If we ignore the radius of the source surface (that is, we assume here that the solar wind flows from the centre of the Sun!) then we can write

$$r = -\frac{U_{sw}}{W_s}(\phi - \phi_0)$$

The angle θ_E that the magnetic field makes with the radial direction at the distance of the Earth can be found from:

$$\theta_E = \tan^{-1}\left(\frac{r_E W_s}{U_{sw}}\right) = \tan^{-1}\left(\frac{1.5\times10^8 \times 2.86\times10^{-6}}{4\times10^2}\right)$$

$$= \tan^{-1}(1.0725) = 47^\circ$$

As given in the diagram on the question sheet, we take the origin of the azimuthal angle ϕ along the Earth-Sun line, so that we have:

$$r_E = \frac{U_{sw}}{W_s}\phi_0$$

, this therefore gives

$$\phi_0 = \frac{r_E W_s}{U_{sw}} = \frac{1.5\times10^8 \times 2.86\times10^{-6}}{4\times10^2}$$

$$= 1.0725 \text{ radians}$$

$$= 61.45^\circ$$