Space Physics Problem Sheet 5

Question 1(Past Exam. Question):
In a magnetised plasma with finite conductivity, such as the Earth's ionosphere, the current density vector $\underline{j}$ can be written as

$$\underline{j} = \sigma_0 \underline{E} - \frac{\sigma_0}{n_e e} \underline{j} \times \underline{B} \quad \cdots \quad (1)$$

where σ_0 is the conductivity of the unmagnetised plasma, n_e is the electron density, and e is the electronic charge and $\underline{E}$, $\underline{B}$ are the electric and magnetic field vectors, respectively.

1a) Show that if the current density vector is $\underline{j} = \underline{j}_{\parallel} + \underline{j}_{\perp}$ where $\underline{j}_{\parallel}$ and $\underline{j}_{\perp}$ are the components of the current density vector parallel and perpendicular, respectively, to the magnetic field vector, then:
$(\underline{j} \times \underline{B}) \times \underline{B} = -B^2 \underline{j}_{\perp}$ where B is the magnitude of the magnetic field.

1b) Taking the vector product of equation (1) with $\underline{B}$, then using the result from part (1a), show that by substituting for $\underline{j} \times \underline{B}$, we get the following expression for the component of the current density vector perpendicular to $\underline{B}$:

$$\underline{j}_{\perp} = \frac{\sigma_0}{1 + \frac{\sigma_0^2 B^2}{n_e^2 e^2}} \underline{E}_{\perp} - \frac{\sigma_0^2}{n_e e} \frac{1}{1 + \frac{\sigma_0^2 B^2}{n_e^2 e^2}} (\underline{E}_{\perp} \times \underline{B}) \quad (2)$$

where $\underline{E}_{\perp}$ is the component of the electric field perpendicular to $\underline{B}$. (Note that we define the component of $\underline{E}$ with respect to the magnetic field as $\underline{E} = \underline{E}_{\parallel} + \underline{E}_{\perp}$.)

1c) The unmagnetised conductivity can be expressed as $\sigma_0 = \dfrac{n_e e^2}{m_e \nu_c}$

where ν_c is the collision frequency of electrons. Given also that the Larmor- or gyrofrequency of the electron in a magnetic field $\underline{B}$ is $\Omega_{ce} = \dfrac{-eB}{m_e}$

show that equation (2) can be put in the form

$$\underline{j}_{\perp} = \frac{\nu_c^2}{\nu_c^2 + \Omega_{ce}^2} \sigma_0 \underline{E}_{\perp} + \frac{\nu_c \Omega_{ce}}{\nu_c^2 + \Omega_{ce}^2} \sigma_0 \frac{\underline{E}_{\perp} \times \underline{B}}{B} \quad (3)$$

1d) Use equation (1) to justify the statement that $\underline{j}_{\parallel} = \sigma_0 \underline{E}_{\parallel}$. Verify that this, together with equation (3) allows the current density vector to be expressed as

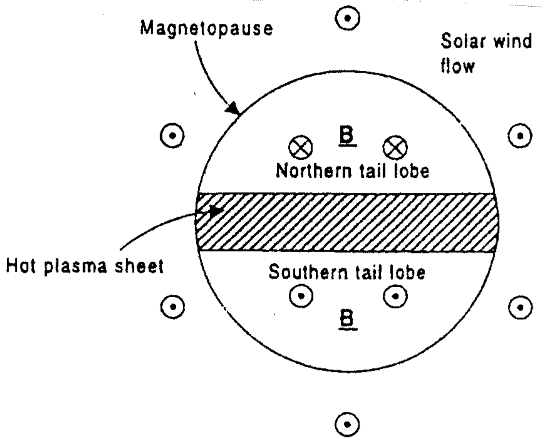
$$\underline{j} = \sigma_0 \underline{E}_{\parallel} + \sigma_P \underline{E}_{\perp} + \sigma_H \frac{\underline{E}_{\perp} \times \underline{B}}{B} \quad (4)$$

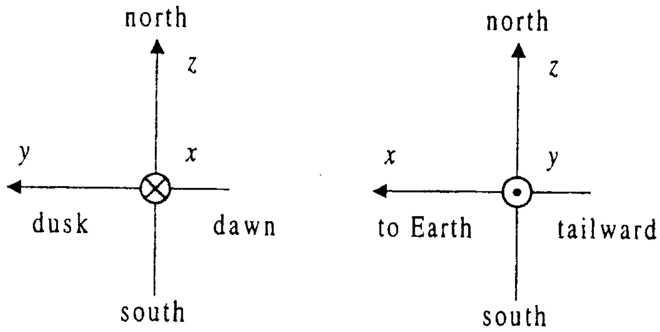
Describe briefly the meaning of this result.

[End of past exam question]

Question 2: A model current sheet – a model for the neutral sheet in the Earth's magnetotail

The simplest analytical model of the neutral sheet which separates the northern and southern lobes of the Earth's magnetotail is the Harris model of the neutral sheet. It assumes magnetic field lines which point towards the Earth in the northern lobe, and away from the Earth in the southern lobe; this corresponds to a stretching out of the Earth's dipole magnetic field in the tail. The co-ordinate system we define has its x- axis pointing towards the Earth in the equatorial plane, the z-axis is pointing northward, and the y-axis completes the system, pointing from dawn to dusk.





Coordinate system used for the neutral sheet in the Earth's magnetotail.

The above figure shows a cross-section of the Earth's magnetotail in the y-z plane, looking towards the Earth and the Sun. In this view the hot plasma sheet constitutes the neutral sheet.

In this system we define the magnetic field vector as $\underline{B}(z) = B_0 \tanh(z/L)\,\hat{\underline{x}}$

and the plasma pressure as $p(z) = p_0 \,\mathrm{sech}^2(z/L)$

where L is a scaling constant (related to a notional thickness of the neutral sheet), B_0 and p_0 are constants. This definition implies that there is no northward component of the magnetic field in the neutral sheet; this is contrary to satellite observations, but the model nevertheless allows a first evaluation of the properties of a neutral sheet.

2a) What is the total (plasma and magnetic) pressure p_{tot} ?

2b) Show that p_{tot} is a constant if $p_0 = B_0^2/2\mu_0$. Calculate the plasma β defined as the ratio of plasma pressure to magnetic field pressure, as a function of z. Comment on the value of β at the centre of the sheet $(z=0)$ and far away from the centre of the sheet $(|z/L| >> 1.)$

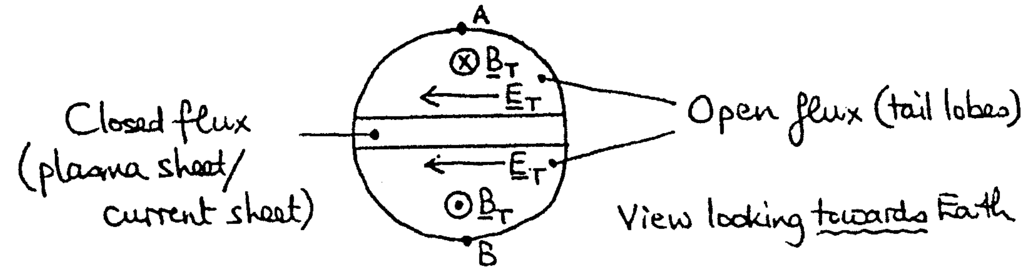
2c) Use Ampere's law to calculate the current density vector $\underline{j}(z/L)$ in the current sheet.

2d) Show that the force balance equation $\underline{j} \times \underline{B} - \nabla p = 0$ is satisfied in this neutral sheet model.

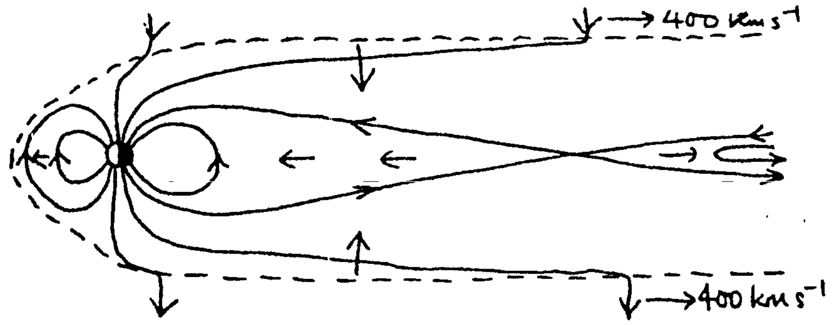
Question 3:

3a) Seen in cross-section, the near-Earth geomagnetic tail consists of two D-shaped "lobes" of open magnetic field lines, back to back, which sandwich a thin layer of closed field lines where the cross-tail current is carried by the plasma sheet population (see diagram overleaf).

The magnetic field strength in the lobes is ~ 20nT, the diameter of the tail is ~ 40 R_E, and the average cross-tail electric field associated with the Dungey flow cycle is $\sim 2 \times 10^{-4}\ Vm^{-1}$. Estimate how long it takes an open flux tube to flow from the northern (or southern) tail lobe magnetopause (point A or B in the diagram) to the current sheet.



3b) The time you have calculated is an estimate of the time for which a given field line remains open in the Dungey cycle (neglecting the short residence time of the open tubes on the dayside of the Earth). During this time the "end" of the open field line threading the magnetopause is carried downstream away from the Earth by the solar wind at $\sim 400\,km\,s^{-1}$ (see diagram below). Hence estimate the length of the geomagnetic tail.



3c) Can you deduce the form of the plasma flow in the ionosphere which is associated with the Dungey (solar wind driven convection) cycle? Start by drawing a circle representing the region of open flux surrounding the north magnetic pole, and decide which directions represent the directions toward and away from the Sun (the noon and midnight meridians). Which way will the field lines move in the region of open flux inside the circle? Which way in the region of closed field lines at lower latitudes (think about the diagrams which we have drawn of flow in the equatorial plane, mapped along field lines into the polar ionosphere)? Complete your diagram by requiring the ionospheric plasma to flow around closed paths which move from regions of closed to open to closed etc field lines.

Question 4:
In this question use the formula derived in lectures which gives the distance of the sub-solar point on the magnetopause from the Earth's (or any magnetised planet's) centre in terms of planetary radii:

$$\frac{R_{MP}}{R_P} = \left(\frac{2 B_{eq}^2}{\mu_0 \rho_{sw} v_{sw}^2}\right)^{1/6}$$

4a) Calculate the distance of the subsolar point on the Earth's magnetopause, using:

$v_{sw} = 400\,km\,s^{-1}$, $n_{sw} = 7\,cm^{-3}$, $B_{eq} = 31{,}000\,nT$

and for the mass of the solar wind particles, use only the proton mass, $m_p = 1.6727 \times 10^{-27}\,kg$.

4b) Calculate R_{MP} for the Earth if v_{sw} is doubled with n_{sw} remaining the same as above.

4c) Calculate the (surface) current density J_{MP} flowing in the magnetopause at the subsolar point for the 2 values of R_{MP}.