Basic Plasmas: Solutions.

(2.11) $n = 5/\text{cm}^3$ $\quad E = \frac{3}{2}\kappa T$

$T_e = 3\,\text{eV}$

so $k_B T_e = 3\times1.6\times10^{-19}\,\text{J} \Rightarrow T_e = \dfrac{3\times1.6\times10^{-19}}{1.38\times10^{-23}} = 3.65\times10^{4}$

ie $T_e = 36{,}500\,\text{K}$ $\quad \times\frac{2}{3}$

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_b T_e}{e^2 n_0}} = \sqrt{\frac{8.85\times10^{-12}\,3\times1.6\times10^{-19}}{(1.6\times10^{-19})^2\,5\times10^{6}}} = \sqrt{3.32\times10^{1}}$$

$= 5.76\,\text{m}$

$N_D = 5\times10^{6}\times(5.76)^3 = 9.55\times10^{8}$

$$\omega_{pe} = \sqrt{\frac{n e^2}{m_e \epsilon_0}} = \sqrt{\frac{5\times10^{6}\,(1.6\times10^{-19})^2}{9.11\times10^{-31}\,8.85\times10^{-12}}} = \sqrt{0.159\times10^{11}}$$

$= 1.26\times10^{5}\,\text{rad/s} \iff f_{pe} = \dfrac{\omega_{pe}}{2\pi} = 2\times10^{4}\,\text{Hz} = 20\,\text{kHz}$

(2.2) $n_e(r) = n_0 e^{-(-e\phi/kT)} \approx n_0\left[1 + \frac{e\phi}{kT}\right]$

so $\rho_q(r) = e(n_p - n_e) \approx e\left(n_0 - n_0\left[1 + \frac{e\phi}{kT}\right]\right) = -\frac{e^2 n_0}{kT}\phi$

Then Poisson's eqn is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{-\rho_q}{\epsilon_0} \stackrel{?}{=} +\frac{e^2 n_0}{\epsilon_0 kT}\phi$$

Try $\phi = \frac{A}{r}e^{-r/\lambda_D}$

$$\Rightarrow \nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(A r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}e^{-r/\lambda_D}\right)\right)$$

$$= \frac{A}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{r^2} - \frac{1}{r\lambda_D}\right)e^{-r/\lambda_D}\right]$$

$$= \frac{A}{r^2}\frac{\partial}{\partial r}\left[\left(-1 - \frac{r}{\lambda_D}\right)e^{-r/\lambda_D}\right]$$

$$= \frac{A}{r^2}\left[\left(0 - \frac{1}{\lambda_D}\right)e^{-r/\lambda_D} + \left(-1 - \frac{r}{\lambda_D}\right)\left(-\frac{1}{\lambda_D}\right)e^{-r/\lambda_D}\right]$$

$$= \frac{A}{r^2}e^{-r/\lambda_D}\left[-\frac{1}{\lambda_D} + \frac{1}{\lambda_D} + \frac{r}{\lambda_D^2}\right] = \frac{A}{r\lambda_D^2}e^{-r/\lambda_D} \stackrel{?}{=} \frac{e^2 n_0}{\epsilon_0 kT}\frac{A}{r}e^{-r/\lambda_D}$$

which is satisfied if $\lambda_D^2 = \frac{\epsilon_0 kT}{n_0 e^2}$ ✓

(3.1) $\Omega = \frac{qB}{m}$ $\qquad R_L = v_\perp / \Omega$

1) 10 keV e^- has $\frac{1}{2}mv^2 = 10^4 \times 1.6 \times 10^{-19}\,J = 1.6\times10^{-15}\,J$

so $v = \sqrt{\frac{2\times1.6\times10^{-15}}{9.11\times10^{-31}}} = \sqrt{0.35\times10^{16}} = 0.6\times10^{8}\,m/s$

$v_\perp = v \sin\alpha = v\sin 45^\circ = 0.42\times10^{8}\,m/s$

$\Omega_e = \frac{1.6\times10^{-19}\; 3\times10^{-5}}{9.11\times10^{-31}} = 0.53\times10^{7} = \underline{5.3\times10^{6}\ rad/s}$

so $R_L = \frac{4.2\times10^{7}}{5.6\times10^{6}} = \underline{7.5\ m}$

2) ~~$\Omega_p = \frac{m_e}{m_p}\ldots = \sqrt{\frac{5.3\times10^{6}}{1843}} = 2.9\times10^{3}$ rad/s~~

$\Omega_p = \frac{1.6\times10^{-19}\; 5\times10^{-9}}{1.67\times10^{-27}} = 4.8\times10^{-1} = 0.48\ rad/s$

$R_L = \frac{v_\perp}{\Omega} = \frac{400\times10^{3}}{0.48} = \underline{833\times10^{3}\,m = 833\ km}$

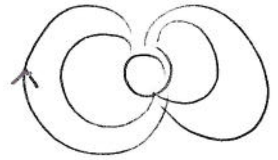
3) 1 keV He^+ $\Rightarrow$ $\frac{1}{2}\,4\times1.6\times10^{-27}\,v^2 = 10^3\times1.6\times10^{-19}$

so $v^2 = 5\times10^{10}$ $\Rightarrow$ $v = 2.2\times10^{5}\,m/s = v_\perp$ $(v_\parallel = 0)$

$\Omega = \frac{1.6\times10^{-19}\; 5\times10^{-2}}{4\times1.6\times10^{-27}} = 1.25\times10^{6}\ rad/s$

Thus $R_L = \frac{v_\perp}{\Omega} = \frac{2.2\times10^{5}}{1.25\times10^{6}} = \underline{0.176\ m}$

(3.2) $B = B_0 \left(\frac{R_e}{r}\right)^3$



∇B drift is at velocity

$$\underset{\sim}{v}_{\nabla B} = \frac{m v_\perp^2}{2 q B^3} \underset{\sim}{B} \times \nabla |\underset{\sim}{B}| = \frac{m v_\perp^2}{2 q B^3} \underset{\sim}{B} \times B_0 (R_e)^3 \left(-\frac{3}{r^4}\right) \hat{r}$$

$$= \frac{m v_\perp^2}{2 q B^3} \frac{3B^2}{r} \left(-\hat{z} \times \hat{r}\right)$$ which is in/out page (depends on sign of q)

so $$|\underset{\sim}{v}_{\nabla B}| = \frac{m v_\perp^2}{2 q B} \frac{3}{r}$$

Now 1 orbit of Earth takes time T s.t. $|v_{\nabla B}| T = 2\pi r$

so $$T = \frac{2\pi r}{|v_{\nabla B}|} = 2\pi r \frac{2 q B r}{3 m v_\perp^2} = \frac{2\pi}{3} \left(\frac{q}{\frac{1}{2} m v_\perp^2}\right) B r^2$$

[Note $\frac{1}{2} m v_\perp^2 / q$ = energy in eV for $|q| = e$]

So e^- & p have same drift period. Putting in numbers.

$$T = \frac{2\pi}{3} \frac{1}{10^3} 0.3 \times 10^{-4} \left(\frac{1}{5}\right)^3 (5 \times 6400 \times 10^3)^2$$

$$= 5.15 \times 10^5 \text{ s} = 143 \text{ hrs} = \underline{6 \text{ days}}$$

a)

$$\underset{\sim}{v}_{gravity} = \frac{\underset{\sim}{E}_{eff} \times \underset{\sim}{B}}{B^2} = \frac{m g^*}{q B} \quad \text{so } T_g = \frac{2\pi r}{m g^*} q B$$

Now $g^* = 9.8 \times \left(\frac{R_e}{r}\right)^2$ so $T_g = \frac{2\pi}{m} \frac{r}{R_e} R_e \left(\frac{r}{R_e}\right)^2 \frac{1}{9.8} q B_0 \left(\frac{R_e}{r}\right)^3$

ie $$T_g = \frac{2\pi}{m} \frac{R_e q B_0}{9.8}$$ (indep. of r!)

so $$T_{ge} = \frac{2\pi}{9.11 \times 10^{-31}} \frac{6400 \times 10^3 \; 1.6 \times 10^{-19} \; 0.3 \times 10^{-4}}{9.8} = 2.2 \times 10^{13} \text{ s}$$

$$= 6.86 \times 10^5 \text{ yrs}$$

$$T_{gp} = T_{ge} \frac{m_e}{m_i} = 1.2 \times 10^{10} \text{ s} = 372 \text{ yrs}$$

3.2 cont.

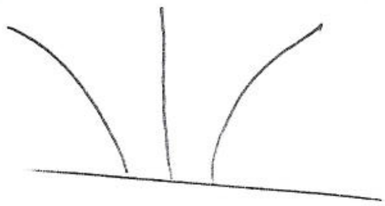
b) Orbital period of uncharged particle is

$$T_{orb} = 2\pi \sqrt{\frac{r^3}{GM_E}}$$

$$= 2\pi \sqrt{\frac{(5 \times 6400 \times 10^3)^3}{6.67 \times 10^{-11}\, 6 \times 10^{24}}} = 8.5 \times 10^4 \, s$$

$$= 1 \text{ day}$$

(3.3) $B = B_0 \frac{H^3}{H^3 + h^3}$ 1)



2) Particles conserve $\mu_m = \frac{m v_\perp^2}{2B}$ and also $\frac{1}{2} m v^2$

so at height h a particle with $(v_{\parallel 0}, v_{\perp 0})$ at $h = 0$ will obey

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 = \frac{1}{2} m v_{\parallel 0}^2 + \frac{1}{2} m v_{\perp 0}^2$$

with $\frac{1}{2} m v_\perp^2 = \frac{1}{2} m v_{\perp 0}^2 \frac{B}{B_0}$

so $\frac{1}{2} m v_\parallel^2 = \frac{1}{2} m v_{\parallel 0}^2 + \frac{1}{2} m v_{\perp 0}^2 \left(1 - \frac{B}{B_0}\right)$

Thus for particles with $v_{\parallel 0} = 0$, they will have

$$\frac{v_\perp^2}{v_\parallel^2} \equiv \tan^2 \alpha = \frac{B/B_0}{1 - B/B_0} = \frac{1}{\frac{B_0}{B} - 1}$$

ie their pitch angle has decreased from 90° (when $\tan\alpha \to \infty$)

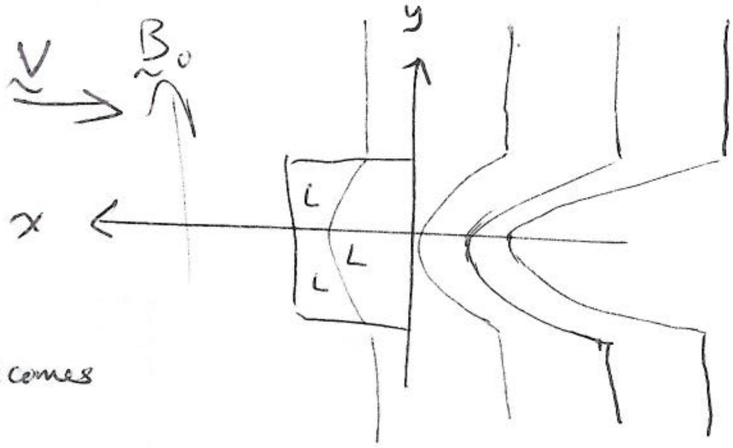
Easy to see all particles decrease α ie

3) so can see this ⇒ beam. At $h \sim 2H$

$$\frac{B_0}{B} = \frac{H^3 + 8H^3}{H^3} = 9 \quad \text{so} \quad \tan^2 \alpha = \frac{1}{8}$$

$$\Rightarrow \alpha = 19^\circ$$

(4.1) $\frac{dV}{dx} = -\left(1-\left|\frac{y}{L}\right|\right)0.9\frac{V_0}{L}$



[Note error in sign of $\frac{dU}{dx}$.
As x increases (to left) V becomes more negative.]

with no deflection of flow can integrate along x:

$$V(x) - V(L) = \int_L^x \frac{dU}{dx}dx = -\frac{V_0}{L}0.9\left(1-\left|\frac{y}{L}\right|\right)\int_L^x dx$$

$$= -\frac{V_0}{L}0.9\left(1-\left|\frac{y}{L}\right|\right)(x-L) \quad \text{with } V(L) = -V_0$$

so $V(x) = -V_0\left[1 - 0.9\left(1-\left|\frac{y}{L}\right|\right)\left(1-\frac{x}{L}\right)\right]$

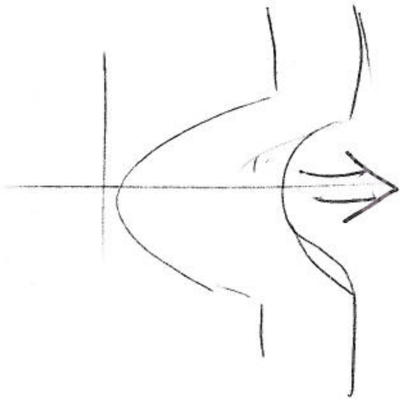
1) Field lines as shown (not obvious they are curved this way, but note at each y V is const downstream so continue to "stretch")

2) From $\nabla \times \underset{\sim}{E} = \underset{\sim}{0}$ + $\hat{z}$ invariant $\Rightarrow E_z = -V_x B_y = \text{const}$

so $V_0 B_0 = V(0)B(0) \Rightarrow B(0,0) = \frac{V_0}{V(0,0)} B_0 = \underline{10\ B_0}$

3) Not obvious. Into compression/heat + transfer to obstacle + increased $B^2/2\mu_0$

4) Kinked $\underset{\sim}{B}$ would try to straighten + accel. flow



(4.3) Waves on string propagate at

$$v_{string} = \sqrt{\frac{\text{Tension}}{\text{mass / length}}}$$

so by analogy $T = B^2/\mu_0$, mass dens $= \rho$

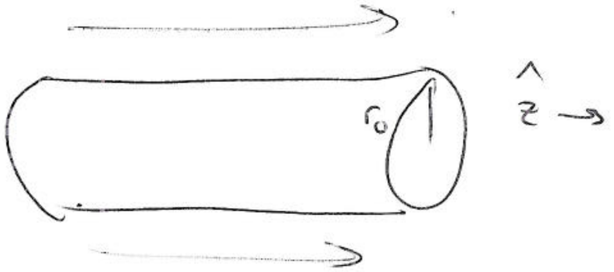
hence $v = \sqrt{\frac{B^2/\mu_0}{\rho}} = v_A$

[Need to think a bit to get this to work.
Consider flux tube of area A. Then total tension (which is "applied" at imaginary end of tube) is $\frac{B^2}{\mu_0} A \equiv T$

$\frac{\text{Mass}}{\text{length}}$ of tube $= \rho A$

$$\left.\right\} \sqrt{\frac{T}{\rho}} = \sqrt{\frac{B^2 A/\mu_0}{\rho A}} = v_A \,]$$

(4.6) $p(r) = p_0\left(1 - \frac{r}{r_0}\right)$



MagnetoHydroStatic (MHS) equilib. in cyl. geom. requires

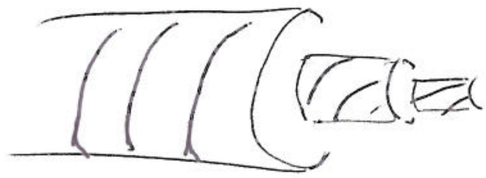
$$\frac{d}{dr}\left(P + \frac{B_\phi^2 + B_z^2}{2\mu_0}\right) + \frac{B_\phi^2}{\mu_0 r} = 0 \qquad \text{Eqn 4.68}$$

Here B_z const so drops out. Try $\frac{B_\phi^2}{2\mu_0} = \frac{r}{3r_0} p_0$

$$\frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0}\right) + \frac{B_\phi^2}{\mu_0 r} = \frac{d}{dr}\left[p_0\left(1 - \frac{r}{r_0}\right) + p_0 \frac{r}{3r_0}\right] + \frac{2}{3}\frac{1}{r_0}p_0$$

$$= \frac{d}{dr}\left[p_0 - \frac{2}{3}p_0\frac{r}{r_0}\right] + \frac{2}{3}\frac{p_0}{r_0} = 0 \checkmark$$

so it <u>is</u> a soln. Note B_ϕ gets larger $\Rightarrow$ more twist for larger r $\longrightarrow$



In cyl. coords with $\frac{\partial}{\partial\phi} = 0 = \frac{\partial}{\partial z}$

$$B_\phi = \sqrt{\frac{2\mu_0}{3r_0} p_0 r}$$

$$\nabla \times \underset{\sim}{B} = 0\hat{r} - \underbrace{\frac{\partial B_z}{\partial r}}_{0}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\hat{z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\sqrt{\frac{2\mu_0 p_0}{3r_0}}\, r^{\frac{1}{2}}\right)\hat{z} = \sqrt{\frac{2\mu_0 p_0}{3r_0}}\,\frac{1}{r}\,\frac{3}{2}\,r^{\frac{1}{2}}\hat{z}$$

so $\underset{\sim}{J} = \frac{1}{\mu_0}\nabla \times \underset{\sim}{B} = \sqrt{\frac{3}{2\mu_0}\frac{p_0}{r_0}}\,\frac{1}{\sqrt{r}}\,\hat{z}$
