Basics - What is a plasma?

- Ionised gas in which dynamics dominated by EM
- Collective behaviour - Debye shielding
- Isolated excess charge $+Q$ attracts cloud of electrons that shield it
  - $\phi = \frac{Q}{4\pi \varepsilon_0 r}$ for a point charge in vacuum
  - $\phi = \frac{Q}{4\pi \varepsilon_0} e^{-r/\lambda_D}$ in plasma
  - $\lambda_D = \sqrt{\frac{e^2 k_B T_e}{\varepsilon_0 n_o}}$
- Need $N_D \equiv n_o \lambda_D^3 \gg 1$ for shielding to work
- Plasmas are overall charge neutral on scale $\gg \lambda_D$
Collective Behaviour - Plasma Oscillations

- Displacements give rise to restoring forces
  - $\omega = \omega_{pe} = \sqrt{\frac{n e^2}{m \epsilon_0}}$
  - Independent of wavelength (cf. $\omega = kc$) so $v_{\text{group}} = 0$
  - Most plasmas are collisionless

Single Particle Motion

Despite the intrinsic collective nature, looking at individual particles is instructive, and also relevant for, e.g., energetic particles.

Lorentz force

$$F = qE + qv \times B$$

Note $B$ does no work on particle, since

$$\frac{d}{dt} \left( \frac{1}{2} m v^2 \right) \equiv m v \cdot \frac{dv}{dt} = v \cdot q (E + v \times B) = qv \cdot E$$

Motion along $B$ is simple:

$$m \frac{dv_{||}}{dt} = qE_{||}$$

Gyromotion

Magnetic force is $\perp$ to $v$ and gives rise to circular motion

Speed is constant, so

$$\frac{mv_{\perp}^2}{R} = qv_{\perp} B$$

which gives the parameters of the (non-relativistic) cyclotron or gyromotion:

Larmor Radius $R_L = \frac{mv_{\perp}}{|q|B}$

Cyclotron Frequency $\Omega_c = \frac{qB}{m}$

Positive particles are left-handed

Gyromotion in action

- Positive particles are left-handed
Helical Motion

So the general motion of a particle in EM fields tends to be

- Motion along $B$ (e.g., constant if $E = 0$ or simple acceleration)
- Gyromotion around magnetic field
- $\Rightarrow$ helical motion

So particles tend to be tied to field lines (keep this thought)

Particle Drifts - $E \times B$

If we add $E_\perp$ to the problem, something very interesting happens:
ALL particles drift $\perp B$ at a velocity $E \times B / B^2$.
Essentially, if we move with this velocity the electric field transforms s.t.
$E_\perp \to 0$ so motion in this frame simple gyromotion.

Can also see this qualitatively by looking at local $R_L$.
Note: Force due to $E$ leads to drift $\perp$ to force!

Particle Drifts - Other forces

For a general force $F$ on a particle, we can think of an effective electric field

$$E_{\text{eff}} \equiv F / q$$

and see immediately that this will lead to a drift

$$v_F = \frac{F \times B}{qB^2}$$

that depends on charge.
Particle Drifts - Curvature and Gradient Drifts

Curvature drift
\[ F_c = \frac{mv_c^2}{R_c \hat{R}_c} \]
\[ v_c = F_c \times B \frac{qB^2}{2} \]

Gradient drift
\[ v_{\nabla B} = \frac{mv_{\perp}^2}{2qB^2} (B \times \nabla |B|) \]

Useful Concepts

Pitch angle (\(\alpha\) = pitch of helix)
\[ \alpha = \tan^{-1} \frac{v_{\perp}}{v_{\parallel}} \]

Guiding Centre (Motion) of centre of gyromotion
\[ \mu_m = \frac{mv_{\perp}^2}{2B^2} = \frac{W_{\perp}}{B^2} \]

\(\mu_m\) is an “Adiabatic Invariant;" it is a constant of the motion in slowly-varying \(L \gg R_L, T \gg 1/\Omega_c\) conditions.

Magnetic Mirroring

Since \(\mu_m = \) constant, if \(B \uparrow\) then \(W_{\perp} \uparrow\).
\(B\) can’t change \(W = W_{\parallel} + W_{\perp}\), so \(W_{\parallel} \downarrow\).
If \(W_{\parallel} \rightarrow 0\), the particle mirrors and can be trapped.

Mirroring in the Magnetosphere
Van Allen Belts

Magnetohydrodynamics - MHD

Basic concepts:
- Treat plasma as fluid (this is simplest collective description)
- Equation of motion of a fluid element
- Overall charge neutral ⇒ no electric forces
- Need equation of state
- Need an Ohm’s Law
- Maxwell’s Equations to describe EM fields ⇒ coupled problem
- If need to know more fine detail, treat as multi-fluid or kinetically
- Can derive fluid description from kinetic theory; need closure assumption

MHD - Fluid equations
Mass conservation (continuity), Newton’s Law, Equation of State:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1}
\]
\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = -\nabla p + j \times \mathbf{B} \tag{2}
\]
\[
\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\rho \rho^{-\gamma}) = 0 \tag{3}
\]

The operator
\[
\frac{d}{dt} = \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \tag{5}
\]
measures rate of change following an element of fluid.

MHD - Maxwell’s Equations
- With charge neutrality, don’t need Gauss’s Law
- \( \nabla \cdot \mathbf{B} = 0 \) will hold, but is just an initial condition on induction equation
- Ignore displacement current (non-relativistic):

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6}
\]
\[
\nabla \times \mathbf{B} = \mu_0 j \tag{7}
\]
\[
\nabla \cdot \mathbf{B} = 0 \tag{8}
\]
- Ohm’s Law for moving conductor:
\[
\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{j}{\sigma} \]
**Induction**
Manipulating Faraday’s Law and Ohm’s Law leads eventually to the MHD Induction Equation

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}
\]

- RHS represents convection and diffusion of \( \mathbf{B} \)
- Magnetic Reynold’s Number

\[
\frac{\text{convection}}{\text{diffusion}} \equiv R_m \equiv \frac{\mathbf{V}B/L}{B/\mu_0 \sigma L^2} = \mu_0 \sigma V L
\]

- \( R_m \ll 1 \) (thin layers, poor conductors) field diffuses and dissipates

\[
\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}
\]

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**Flux Freezing**

\( R_m \gg 1 \) plasma and magnetic flux are “frozen” to one another

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**Applications of Flux Freezing - Comets**

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**Applications of Flux Freezing - Current Sheets and Boundary Layers**
Applications of Flux Freezing - Cell Model of Plasma Universe

Magnetic Forces
- Substitute for $j$ from Ampere’s Law
- A lot of manipulation
- $j \times B = -\nabla_\perp \left( \frac{B^2}{2\mu_0} \right) - \frac{B^2 R_c}{\mu_0 R_c^2}$
  - $\Rightarrow$ magnetic forces act like a (2D) pressure in directions perpendicular to $B$
  - Plus tension (straightening curves) along $B$.
  - Equilibria are balance of magnetic and thermal pressure gradient forces
  - Not all equilibria are stable!

MHD Waves
- Waves illustrate forces in a medium
- Waves transport energy and momentum
- Waves can accelerate (or decelerate) energetic particles
- Waves are fun

Alfvén Waves
- “Magnetic Tension” suggests waves on a string
- Speed of propagation $v_A = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{B^2}{\mu_0\rho}}$
- $\omega = \pm k_\parallel v_A$
- $N_B v_{\text{group}} = v_A \hat{B}_0$
- $\delta V_{v_A} = \pm \delta B \hat{B}_0$
- $\delta \rho = 0$
Magnetosonic Waves

- “Magnetic Pressure” can work with thermal pressure
- Speed of propagation something like $v_{ms} = \sqrt{\gamma P/\rho}$
- Here $\gamma P = \gamma P_{\text{thermal}} + 2P_{\text{magnetic}}$
- 2 because $B$ pressure is 2D
- So $v_{ms}^2 = (\gamma P/\rho) + (2B^2/2\mu_0\rho)$
- i.e., $v_{ms}^2 = c_s^2 + v_A^2$
- Above ok for $k \perp B_0$
- Can also get slow mode, in which thermal and magnetic compete

General Propagation

$v_A < c_s$

$v_A > c_s$

What have we left out?

- Kinetics
  - Collisions
  - Non-thermal particles & acceleration
  - Microinstabilities & anomalous transport
- Discontinuities (Shocks, TD’s, RD’s)
- Dynamos
- Relativity
- Magnetic Reconnection
- Real plasmas
  - Solar wind, magnetosphere
  - Solar atmosphere
  - Partially ionised media (ionosphere)
- Multi-fluid
- Simulations, data analysis techniques
- and a whole lot more!